Elements of Computer-Algebraic Analysis

Viktor Levandovskyy
Lehrstuhl D für Mathematik, RWTH Aachen, Germany

22.07. Tutorial at ISSAC 2012, Grenoble, France

Recommended literature (textbooks)

1. J. C. McConnell, J. C. Robson, "Noncommutative Noetherian Rings", Graduate Studies in Mathematics, 30, AMS (2001)
2. G. R. Krause and T. H. Lenagan, "Growth of Algebras and Gelfand-Kirillov Dimension", Graduate Studies in Mathematics, 22, AMS (2000)
3. S. Saito, B. Sturmfels and N. Takayama, "Gröbner Deformations of Hypergeometric Differential Equations", Springer, 2000
4. J. Bueso, J. Gómez–Torrecillas and A. Verschoren, "Algorithmic methods in non-commutative algebra. Applications to quantum groups", Kluwer , 2003
5. H. Kredel, "Solvable polynomial rings", Shaker Verlag, 1993
6. H. Li, "Noncommutative Gröbner bases and filtered-graded transfer", Springer, 2002

Recommended literature (textbooks and PhD theses)

1. V. Ufnarovski, "Combinatorial and Asymptotic Methods of Algebra", Springer, Encyclopedia of Mathematical Sciences 57 (1995)
2. F. Chyzak, "Fonctions holonomes en calcul formel", PhD. Thesis, INRIA, 1998
3. V. Levandovskyy, "Non-commutative Computer Algebra for polynomial algebras: Gröbner bases, applications and implementation" PhD. Thesis, TU Kaiserslautern, 2005
4. C. Koutschan, "Advanced Applications of the Holonomic Systems Approach", PhD. Thesis, RISC Linz, 2009
5. K. Schindelar, "Algorithmic aspects of algebraic system theory", PhD. Thesis, RWTH Aachen, 2010

Software

D-modules and algebraic analysis:

- KAN/SM1 by N. Takayama et al.
- `D-modules` package in MACAULAY2 by A. Leykin and H. Tsai
- RISA/ASIR by M. Noro et al.
- OREMODULES package suite for MAPLE by D. Robertz, A. Quadrat et al.
- SINGULAR:PLURAL with a D-module suite; by V. L. et al.

holonomic and D-finite functions:

- GROEBNER, ORE ALGEBRA, MGFUN, ... by F. Chyzak
- HOLONOMICFUNCTIONS by C. Koutschan
- SINGULAR:LOCAPAL (partly under development) by V. L. et al.

Overview

- Operator algebras and their partial classification
- More general: G-algebras and Gröbner bases in G-algebras
- Module theory; Dimension theory; Gel'fand-Kirillov dimension
- Linear modeling with variable coefficients
- Elimination of variables and Gel'fand-Kirillov dimension
- Ore localization; smallest Ore localizations
- Solutions via homological algebra
- The complete annihilator program
- Some computational D-module theory, Weyl closure
- Purity; pure modules, pure functions, preservation of purity
- Purity filtration of a module; connection to solutions
- Jacobson normal form

What is computer algebraic Analysis?

Algebraic Analysis

1. As a notion, it arose in 1958 in the group of Mikio Sato (Japan)
2. Main objects: systems of linear partial DEs with variable coefficients, generalized functions
3. Main idea: study systems and generalized functions in a coordinate-free way (i. e. by using modules, sheaves, categories, localizations, homological algebra, . . .)
4. Keywords include D-Modules, (sub-)holonomic D-Modules, regular resp. irregular holonomic D-Modules
5. Interplay: singularity theory, special functions, ...

Other ingredients: symbolic algorithmic methods for discrete resp. continuous problems like symbolic summation, symbolic integration etc.

What is computer-algebraic Analysis?

Algebraization as a trend

Algebra: Ideas, Concepts, Methods, Abstractions

Computer algebra works with algebraic concepts in a (semi-)algorithmic way at three levels:

1. Theory: Methods of Algebra in a constructive way
2. Algorithmics: Algorithms (or procedures) and their Correctness, Termination and Complexity results (if possible)
3. Realization: Implementation, Testing, Benchmarking, Challenges; Distribution, Lifecycle, Support and software-technical aspects

Some important names in computer-algebraic analysis

- W. Gröbner and B. Buchberger: Gröbner bases and constructive ideal/module theory
- O. Ore: Ore Extension and Ore Localization
- I. M. Gel'fand and A. Kirillov: GK-Dimension
- B. Malgrange: M. isomorphism, M. ideal, . . .
- J. Bernstein, M. Sato, M. Kashiwara, C. Sabbah, Z. Mebkhout, B. Malgrange et al.: D-module theory
- N. Takayama, T. Oaku, B. Sturmfels, M. Saito, M. Granger, U. Walther, F. Castro, H. Tsai, A. Leykin et al.: (not only) computational D-module theory
- . . .

Part I. Operator algebras and their partial classification.

Operator algebras: partial Classification

Let K be an effective field, that is $(+, -, \cdot, :)$ can be performed algorithmically.
Moreover, let $\mathcal{F}$ be a K-vector space ("function space").

Let x be a local coordinate in $\mathcal{F}$. It induces a K-linear map $X : \mathcal{F} \to \mathcal{F}$, i. e. $X(f) = x \cdot f$ for $f \in \mathcal{F}$. Moreover, let

$\mathfrak{o}_x : \mathcal{F} \to \mathcal{F}$ be a K-linear map.

Then, in general, $\mathfrak{o}_x \circ X \neq X \circ \mathfrak{o}_x$, that is $\mathfrak{o}_x(x \cdot f) \neq x \cdot \mathfrak{o}_x(f)$ for $f \in \mathcal{F}$.

The **non-commutative relation** between $\mathfrak{o}_x$ and X can be often read off by analyzing the properties of $\mathfrak{o}_x$ like, for instance, the product rule.

Classical examples: Weyl algebra

Let $f : \mathbb{C} \to \mathbb{C}$ be a differentiable function and $\partial(f(x)) := \frac{\partial f}{\partial x}$.
Product rule tells us that $\partial(x\ f(x)) = x\ \partial(f(x)) + f(x)$, what is translated into the following relation between operators

$$(\partial \ \circ \ x - x \ \circ \ \partial - 1) \ (f(x)) = 0.$$

The corresponding operator algebra is the 1st **Weyl algebra**

$$D_1 \ = \ K\langle x, \partial \mid \partial x = x\partial + 1\rangle.$$

Classical examples: shift algebra

Let g be a sequence in discrete argument k and $\mathbf{s}$ is the shift operator $\mathbf{s}(g(k)) = g(k+1)$. Note, that $\mathbf{s}$ is multiplicative.

Thus $\mathbf{s}(kg(k)) = (k+1)g(k+1) = (k+1)\mathbf{s}(g(k))$ holds.

The operator algebra, corr. to $\mathbf{s}$ is the 1st **shift algebra**

$$S_1 \ = \ K\langle k, s \mid sk = (k+1)s = ks + s\rangle.$$

Intermezzo

For a function in differentiable argument x and in discrete argument k the natural operator algebra is

$$A = D_1 \otimes_K S_1 = K\langle x, k, \partial_x, s_k \mid \partial_x x = x\partial_x + 1,\ s_k k = ks_k + s_k,$$
$$xk = kx,\ xs_k = s_k x,\ \partial_x k = k\partial_x, \partial_x s_k = s_k \partial_x\rangle.$$

Examples form the q-World

Let $k \subset K$ be fields and $q \in K^*$.
In q-calculus and in quantum algebra three situations are common for a fixed k: (a) $q \in k$, (b) q is a root of unity over k, and (c) q is transcendental over k and $k(q) \subseteq K$.

Let $\partial_q(f(x)) = \frac{f(qx)-f(x)}{(q-1)x}$ be a q-differential operator.
The corr. operator algebra is the 1st q-**Weyl algebra**

$$D_1^{(q)} = K\langle x, \partial_q \mid \partial_q x = q \cdot x\partial_q + 1\rangle.$$

The 1st q-**shift algebra** corresponds to the q-shift operator $\mathbf{s}_q(f(x)) = f(qx)$:

$$K_q[x, s_q] = K\langle x, s_q \mid s_q x = q \cdot x s_q\rangle.$$

Two frameworks for bivariate operator algebras

Algebra with linear (affine) relation

Let $q \in K^*$ and $\alpha, \beta, \gamma \in K$. Define

$$\mathcal{A}^{(1)}(q, \alpha, \beta, \gamma) := K\langle x, y \mid yx - q \cdot xy = \alpha x + \beta y + \gamma\rangle$$

Because of **integration operator** $\mathcal{I}(f(x)) := \int_a^x f(t)dt$ for $a \in \mathbb{R}$, obeying the relation $\mathcal{I}\ x - x\ \mathcal{I} = -\mathcal{I}^2$ we need yet more general framework.

Algebra with nonlinear relation

Let $N \in \mathbb{N}$ and $c_0, \ldots, c_N, \alpha \in K$. Then $\mathcal{A}^{(2)}(q, c_0, \ldots, c_N, \alpha)$ is
$K\langle x, y \mid yx - q \cdot xy = \sum_{i=1}^n c_i y^i + \alpha x + c_0\rangle$ or
$K\langle x, y \mid yx - q \cdot xy = \sum_{i=1}^n c_i x^i + \alpha y + c_0\rangle$.

Theorem (L.–Koutschan–Motsak, 2011)

*$\mathcal{A}^{(1)}(q, \alpha, \beta, \gamma) = K\langle x, y \mid yx - q \cdot xy = \alpha x + \beta y + \gamma\rangle$,
where $q \in K^*$ and $\alpha, \beta, \gamma \in K$
is isomorphic to the 5 following* **model algebras**:

1. $K[x, y]$,
2. *the 1st Weyl algebra* $D_1 = K\langle x, \partial \mid \partial x = x\partial + 1\rangle$,
3. *the 1st shift algebra* $S_1 = K\langle x, s \mid sx = xs + s\rangle$,
4. *the 1st q-commutative algebra* $K_q[x, s] = K\langle x, s \mid sx = q \cdot xs\rangle$,
5. *the 1st q-Weyl algebra* $D_1^{(q)} = K\langle x, \partial \mid \partial x = q \cdot x\partial + 1\rangle$.

Theorem (L.–Makedonsky–Petravchuk, new)

*For $N \geq 2$ and $c_0, \ldots, c_N, \alpha \in K$, $\mathcal{A}^{(2)}(q, c_0, \ldots, c_N, \alpha)$
$= K\langle x, y \mid yx - q \cdot xy = \sum_{i=1}^N c_i y^i + \alpha x + c_0\rangle$ is isomorphic to ...*

1. $K_q[x, s]$ *or* $D_1^{(q)}$, **if** $q \neq 1$,
2. $S_1 = K\langle x, s \mid sx = xs + s\rangle$, **if** $q = 1$ **and** $\alpha \neq 0$,
3. $K\langle x, y \mid yx = xy + f(y)\rangle$, *where* $f \in K[y]$ *with* $\deg(f) = N$, **if** $q = 1$ **and** $\alpha = 0$.

Application

Given a system of equations S in terms of other operators,

one can look up a concrete isomorphism of K-algebras (e. g. from the mentioned papers)

and rewrite S as S' in terms of the operators above.

Further results on S' after performing computations can be transferred back to original operators.

Example: difference and divided difference operators $\Delta_n = S_n - 1$, $\Delta_n^{(q)} = S_n^{(q)} - 1$ etc.

Quadratic algebras

Lemma (L.–Makedonsky–Petravchuk, new)

$K\langle x, y \mid yx = xy + f(y)\rangle \cong K\langle z, w \mid wz = zw + g(w)\rangle$

if and only if

$\exists \lambda, \nu \in K^*$ and $\exists \mu \in K$, such that $g(t) = \nu f(\lambda t + \mu)$ (in particular $\deg(f) = \deg(g)$).

Lemma (L.–Makedonsky–Petravchuk, new)

For any algebra of the type $B = K\langle a, b \mid ba = ab + f(a)\rangle$ for $f \neq 0$ there exists an injective homomorphism into the 1st Weyl algebra.

Quadratic algebras

Let $N = \deg f(y) = 2$ and K be algebraicaly closed field of char $K > 2$. Then there are precisely two classes of non-isomorphic algebras of the type $K\langle x, y \mid yx = xy + f(y)\rangle$:

$K\langle x, y \mid yx = xy + y^2\rangle$ type

- integration algebra $K\langle x, \mathcal{I} \mid \mathcal{I}\, x = x\, \mathcal{I} - \mathcal{I}^2\rangle$,
- the algebra $K\langle x^{-1}, \partial = \frac{d}{dx} \mid \partial x^{-1} = x^{-1}\partial - (x^{-1})^2\rangle$,
- the algebra $K\langle x, \partial^{-1} \mid \partial^{-1} x = x\partial^{-1} - (\partial^{-1})^2\rangle$ etc.

$K\langle x, y \mid yx = xy + y^2 + 1\rangle$ type

- tangent algebra $K\langle \tan, \partial \mid \partial \cdot \tan = \tan \cdot \partial + \tan^2 + 1\rangle$ (take $y = \tan$, $x = -\partial$)
- the subalgebra of the 1st Weyl algebra, generated by $Y = -x$ and $X = (x^2 + 1)\partial$; then $YX = XY + Y^2 + 1$ etc.

Open problems for the Part 1

Let A be a bivariate algebra as before.

- If S is a multiplicatively closed Ore set (see next parts), then there exists localization $S^{-1}A$, such that $A \subset S^{-1}A$ holds.
- Problem: establish isomorphy classes for the localized algebras $S^{-1}A$, depending on the type of S.
- Example: in the part on localization.

More general framework: G-algebras

Let $R = K[x_1, \ldots, x_n]$. The standard **monomials** $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, $\alpha_i \in \mathbb{N}$, form a K-basis of R.

$$\mathrm{Mon}(R) \ni x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \mapsto (\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha \in \mathbb{N}^n.$$

1. a total ordering $\prec$ on $\mathbb{N}^n$ is called a **well–ordering**, if $\forall F \subseteq \mathbb{N}^n$ there exists a minimal element of F, in particular $\forall\, a \in \mathbb{N}^n$, $0 \prec a$
2. an ordering $\prec$ is called a **monomial ordering on** R, if
   - $\forall \alpha, \beta \in \mathbb{N}^n\ \alpha \prec \beta \ \Rightarrow \ x^\alpha \prec x^\beta$
   - $\forall \alpha, \beta, \gamma \in \mathbb{N}^n$ such that $x^\alpha \prec x^\beta$ we have $x^{\alpha+\gamma} \prec x^{\beta+\gamma}$.
3. Any $f \in R \setminus \{0\}$ can be written uniquely as $f = cx^\alpha + f'$, with $c \in K^*$ and $x^{\alpha'} \prec x^\alpha$ for any non–zero term $c' x^{\alpha'}$ of f'.
   $\mathrm{lm}(f) = x^\alpha$, the **leading monomial** of f
   $\mathrm{lc}(f) = c$, the **leading coefficient** of f.

Towards G-algebras

Suppose we are given the following data

1. a field K and a commutative ring $R = K[x_1, \ldots, x_n]$,
2. a set $C = \{c_{ij}\} \subset K^*$, $1 \leq i < j \leq n$
3. a set $D = \{d_{ij}\} \subset R$, $\quad 1 \leq i < j \leq n$

Assume, that there is a monomial well–ordering $\prec$ on R such that

$$\forall 1 \leq i < j \leq n, \ \mathrm{lm}(d_{ij}) \prec x_i x_j.$$

To the data $(R, C, D, \prec)$ we associate an algebra

$$A = K\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{ij} \cdot x_i x_j + d_{ij}\}\ \forall 1 \leq i < j \leq n\rangle.$$

A is called **a** G**–algebra** in n variables, if

$$c_{ik} c_{jk} \cdot d_{ij} x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} \cdot d_{ik} x_j + d_{jk} x_i - c_{ij} c_{ik} \cdot x_i d_{jk} = 0.$$

G-algebras

Theorem (Properties of G-algebras)

Let A be a G-algebra in n variables. Then

- *A is left and right Noetherian,*
- *A is an integral domain,*
- *the Gel'fand-Kirillov dimension over K is* $\mathrm{GKdim}(A) = n$,
- *the global homological dimension* $\mathrm{gl.dim}(A) \leq n$,
- *the generalized Krull dimension* $\mathrm{Kr.dim}(A) \leq n$.
- *A is Auslander-regular and a Cohen-Macaulay algebra.*

Classical examples: full shift algebra

Adjoining the backwards shift $s^{-1} : f(x) \mapsto f(x-1)$ to the shift algebra, we incorporate several more relations, which define a so-called **full shift algebra**:

$$K\langle x, s, s^{-1} \mid sx = (x+1)s,\ s^{-1}x = (x-1)s^{-1}, s^{-1}s = s \cdot s^{-1} = 1\rangle$$

Note: full shift algebra is **not** a G-algebra, due to the relation $s \cdot s^{-1} = 1$. But it can be realized as a factor algebra of a G-algebra
$A = K\langle x, s, s^{-1} \mid sx = (x+1)s,\ s^{-1}x = (x-1)s^{-1}, s^{-1}s = ss^{-1}\rangle$
modulo the two-sided ideal $\langle s^{-1}s - 1\rangle$.

We can also realize this algebra as an Ore localization of the shift algebra, see next parts.

Gröbner Bases in G-algebras

Let A be a G-algebra in $x_1, \dots, x_n$. From now on, we assume that a given ordering is a **well-ordering**.

Definition

We say that $x^\alpha \mid x^\beta$, i. e. monomial x^α **divides** monomial x^β, if $\alpha_i \le \beta_i \; \forall i = 1 \dots n$.

It means that x^β is **reducible** by x^α, that is there exists $\gamma \in \mathbb{N}^n$, such that $\beta = \alpha + \gamma$. Then $\mathrm{lm}(x^\alpha x^\gamma) = x^\beta$, hence $x^\alpha x^\gamma = c_{\alpha\gamma} x^\beta +$ lower order terms.

Definition

Let $\prec$ be a monomial ordering on A, $I \subset A$ be a left ideal and $G \subset I$ be a finite subset. G is called a **(left) Gröbner basis** of I, if $\forall f \in I \setminus \{0\}$ there exists a $g \in G$ satisfying $\mathrm{lm}(g) \mid \mathrm{lm}(f)$.

Gröbner Bases in G-algebras

- There exists a generalized Buchberger's algorithm (as well as other generalized algorithms for Gröbner bases), which works along the lines of the classical commutative algorithm.
- There exist algorithms for computing a two-sided Gröbner basis, which has no analogon in the commutative case.
- G-algebras are fully implemented in the actual system SINGULAR:PLURAL, as well as in older systems MAS, FELIX.
- In SINGULAR:PLURAL there are many thorougly implemented functions, including Gröbner bases, Gröbner basics (module arithmetics) and numerous useful tools.

Gröbner Technology = Gröbner trinity + Gröbner basics

Gröbner trinity:

- left Gröbner basis of a submodule of a free module
- left syzygy module of a given set of generators
- left transformation matrix, expressing elements of Gröbner basis in terms of original generators

Gröbner basics (Buchberger, Sturmfels, ...)

- Ideal (resp. module) membership problem (NF, REDUCE)
- Intersection with subrings (ELIMINATE)
- Intersection and quotient of ideals (INTERSECT, QUOT)
- Kernel of a module homomorphism (MODULO)
- Kernel of a ring homomorphism (PREIMAGE)
- Algebraic dependencies of commuting polynomials
- Hilbert polynomial of graded ideals and modules ...

Part II. Dimension theory.

From system of equations to modules

Consider Legendre's differential equation (order 2 in ∂_x)

$$(x^2-1)P''_n(x)^2 + 2xP'_n(x) - n(1+n)P_n(x) = 0$$

- x is differentiable with ∂_x as corr. operator
- if $n \in \mathbb{Z}$, n is discretely shiftable with s_n as corr. op.
- then there is a recursive formula of Bonnet (order 2 in shift s_n)

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$$

$$\mathfrak{D} := K\langle n, s_n \mid s_n n = ns_n + s_n\rangle \otimes_K K\langle x, \partial_x \mid \partial_x x = x\partial_x + 1\rangle.$$

From the system of equations

$$\begin{aligned} (x^2-1)P''_n(x)^2 + 2xP'_n(x) - n(1+n)P_n(x) &= 0, \\ (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) &= 0. \end{aligned}$$

one obtains the matrix $P \in \mathfrak{D}^{2\times 1}$; thus $M = \mathfrak{D}/\mathfrak{D}^{1\times 2}P$ and

$$\begin{bmatrix} (x^2-1)\partial_x^2 + 2x\partial_x - n(1+n) \\ (n+2)s_n^2 - (2n+3)xs_n + n + 1 \end{bmatrix} \bullet P_n(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

With the help of Gröbner bases over $\mathfrak{D}$: a minimal generating set of the left ideal P contains a *compatibility condition*

$$(n+1)s_n\partial_x - (n+1)x\partial_x - (n+1)^2 \equiv (n+1)(s_n\partial_x - x\partial_x + n + 1).$$

From system of equations to modules

Let $f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)$ be unknown generalized functions, for instance from $C^\infty(\mathbb{R}^n)$.
Then a homogeneous system of linear functional (operator) equations with coefficients from $K[x_1,\ldots,x_n]$ can be presented via the matrix equation in the corresponding operator algebra $\mathfrak{D}$:

$$P \cdot \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad P \in \mathfrak{D}^{\ell\times m}$$

One associates to the system a left $\mathfrak{D}$-module $M = \mathfrak{D}^{1\times m}/\mathfrak{D}^{1\times\ell}P$, saying M **is finitely presented by a matrix** P.

From system of equations to modules

Different matrices P_i can represent the same module M.

For instance, for any unimodular $T \in \mathfrak{D}^{\ell\times\ell}$ one has
$Pf = 0 \Leftrightarrow (TP)f = 0$ and also $\mathfrak{D}^{1\times m}/\mathfrak{D}^{1\times\ell}TP \cong \mathfrak{D}^{1\times m}/\mathfrak{D}^{1\times\ell}P$.

For various purposes we might utilize different presentations of M. The invariants of a module M, like dimensions, do not depend on the presentation.

Algebraic manipulations from the left on P often need algorithms for left Gröbner bases for a submodule of a free module, generated by rows or columns of P (thus not only GBs of ideals).

From modules to solutions of systems

Let $\mathcal{F}$ be a left $\mathfrak{D}$-module (not necessarily finitely presented), and P a system of equations as before, then

$$\mathrm{Sol}_{\mathfrak{D}}(P,\mathcal{F}) := \{f \in \mathcal{F}^{m\times 1} \ : \ P \bullet f = 0\}.$$

Noether-Malgrange Isomorphism

There exists an isomorphism of K-vector spaces

$$\mathrm{Hom}_{\mathfrak{D}}(M,\mathcal{F}) = \mathrm{Hom}_{\mathfrak{D}}(\mathfrak{D}^{1\times m}/\mathfrak{D}^{1\times \ell}P,\mathcal{F}) \ \cong \ \mathrm{Sol}_{\mathfrak{D}}(P,\mathcal{F}),$$

$$(\phi : M \to \mathcal{F}) \ \mapsto (\phi([e_1]),\ldots,\phi([e_m])) \in \mathcal{F}^{m\times 1}.$$

From functions to modules

Let $\mathcal{F}$ be a left $\mathfrak{D}$-module (not necessarily finitely presented), and $f \in \mathcal{F}$. Consider $\mathfrak{D}f = \{\mathfrak{o} \bullet f \mid \mathfrak{o} \in \mathfrak{D}\}$, which is an $\mathfrak{D}$-submodule of $\mathcal{F}$.

Consider a homomorphism of left $\mathfrak{D}$-modules $\phi_f : \mathfrak{D} \to \mathcal{F}$, $\mathfrak{o} \mapsto \mathfrak{o} \bullet f$, in other words $\phi_f(1) = f \in \mathcal{F}$. Then

- $Im\phi_f = \mathfrak{D}f$, $\ \mathrm{Ker}\,\phi_f = \{\mathfrak{o} \in \mathfrak{D} : \mathfrak{o} \bullet f = 0\} =: \mathrm{Ann}_{\mathfrak{D}}\,\mathbf{f}$
- as left $\mathfrak{D}$-modules, one has $\ \mathfrak{D}f \cong \mathfrak{D}/\,\mathrm{Ann}_{\mathfrak{D}}\, f$
- hence $\mathfrak{D}f$ is finitely presented left $\mathfrak{D}$-module.

An element $m \in \mathcal{F}$ is called a **torsion element**, if $\mathrm{Ann}_{\mathfrak{D}}\, m \neq 0$.

Many classical functions in common functional spaces are torsion.

Hence, algorithms for the computation of the left ideal $\mathrm{Ann}_{\mathfrak{D}}\, m$ (which is finitely generated when $\mathfrak{D}$ is Noetherian) are very important.

From functions to modules

Many classical functions in common functional spaces are torsion. **But not all**.

Example: $f = \tan(x)$ is not a torsion element in a module over Weyl algebra, since there exists **no** system of linear ODEs with variable coefficients, having $\tan(x)$ as solution. However, there is a nonlinear ODE $f' = 1 + f^2$.

Recall: we are able to treat polynomials in the operator $\tan(x)\cdot$ as coefficients in an algebra with differentiation w.r.t x.

From functions to modules

Let $\mathcal{F}$ be a left $\mathfrak{D}$-module, and $f_1,\ldots,f_m \in \mathcal{F}$ be torsion elements. Consider $M = \mathfrak{D}f_1 + \ldots + \mathfrak{D}f_m$. As we know, every $\mathfrak{D}f_i$ is finitely presented $\mathfrak{D}$-submodule of $\mathcal{F}$.

Consider a homomorphism of left $\mathfrak{D}$-modules

$$\phi : \mathfrak{D}^m = \bigoplus_{i=1}^{m} \mathfrak{D}e_i \to \mathcal{F}, \quad \sum \mathfrak{o}_i e_i \mapsto \sum \mathfrak{o}_i \bullet f_i,$$

in other words $\phi(e_i) = f_i \in \mathcal{F}$. Then $Im\ \phi = M = \sum \mathfrak{D}f_i$,

- $\mathrm{Ker}\,\phi = \{[\mathfrak{o}_1,\ldots,\mathfrak{o}_m] \in \mathfrak{D}^m : \sum \mathfrak{o}_i \bullet f_i = 0\} =: \mathrm{Mann}_{\mathfrak{D}}\,\mathbf{M}$
- as left $\mathfrak{D}$-modules, one has $\ M = \sum \mathfrak{D}f_i \cong \mathfrak{D}^m/\,\mathrm{Mann}_{\mathfrak{D}}\, M$
- hence $M = \sum_i \mathfrak{D}f_i$ is finitely presented left $\mathfrak{D}$-module.

Clearly $\oplus\,\mathrm{Ker}\,\phi_{f_i} e_i \subseteq \mathrm{Mann}_{\mathfrak{D}}\, M$.

Idea: Model polynomial-exponential signals by linear systems.
Question: What is more precise in such a modeling: operator algebras with constant or with polynomial coefficients?

Answer: algebras with polynomial coefficients.

Theorem (Zerz–L.–Schindelar, 2011)

Let $K = \mathbb{R}$, $p_i \in K[x_1, \ldots, x_n]^\ell$ and $V = Kp_1 + \cdots + Kp_m$. Let $\mathfrak{D}$ be the n-th Weyl algebra and $\mathfrak{D} \supset \mathrm{Ann}_{\mathfrak{D}}(V) := \cap \mathrm{Ann}_{\mathfrak{D}}\, p_i$ be the left ideal of operators, simultaneously annihinalting $p_1, \ldots, p_m$. Then

$$\mathrm{Sol}_{\mathfrak{D}}(\,\mathfrak{D}/\mathrm{Ann}_{\mathfrak{D}}(V),\; C^\infty(\mathbb{R}^\ell)) = V.$$

Keywords: **V**ariant **M**ost **P**owerful **U**nfalsified **M**odel, cf. two recent papers by Zerz, L. and Schindelar.

Dimensions

- Generalized Krull dimension (for an algebra or a module, $\mathrm{Kr.dim}\, M$) is called Krull-Rentschler-Gabriel dimension; not algorithmic
- projective dimension of a module, $\mathrm{p.dim}\, M$; algorithmic (relatively expensive), implemented
- global homological dimension of an algebra, $\mathrm{gl.dim}\, A = \sup\{\mathrm{p.dim}\, M : M \in A - mod\}$, in general not algorithmic
- homological grade of a module, $j(M)$; algorithmic (a little less expensive than $\mathrm{p.dim}\, M$), implemented
- Gel'fand-Kirillov Dimension; algorithmic (relatively cheap), implemented; intuition: similar to usual Krull dimension

Filtration on algebras and modules

Let A be a K-algebra, generated by $x_1, \ldots, x_m$.

Degree filtration

Let $V = Kx_1 \oplus \ldots \oplus Kx_m$ be a vector space.

Set $V_0 = K$, $V_1 = K \oplus V$ and $V_{k+1} = V_k \oplus V^{k+1}$. If

$$V_i \subseteq V_{i+k}, \quad V_i \cdot V_j \subseteq V_{i+j}, \quad A = \bigcup_{k=0}^{\infty} V_k,$$

then $\{V_k \mid k \in \mathbb{N}\}$ is the **standard (ascending) filtration** of A.

Gel'fand-Kirillov dimension and its properties

Let $M_0 \subset M$ be a finite K-vector space, spanned by the generators of M. That is $\dim_K M_0 < \infty$ and $AM_0 = M$.

$\{H_d := V_d M_0, d \in \mathbb{N}\}$ is an induced ascending filtration on M.

The **Gel'fand-Kirillov dimension** of M is defined as follows

$$\mathrm{GKdim}(M) \;=\; \limsup_{d\to\infty} (\log_d(\dim_K H_d))$$

In the standard construction one puts $\deg x_i := 1$ and defines $V_d := \{f \mid \deg f = d\}$ and $V^d := \{f \mid \deg f \le d\}$.

Conventions: $\mathrm{GKdim}(0) = -\infty$. $\mathrm{GKdim}_{\mathbb{Q}}(\mathbb{Q}) = 0$.

Lemma

Let A be a K-algebra and a domain. If the standard filtration on A is compatible with the PBW Basis $\{x^\alpha \mid \alpha \in \mathbb{N}^m\}$, then $\mathrm{GKdim}_K(A) = m$.

$$\dim V_d = \binom{d+m-1}{m-1},\ \dim V^d = \binom{d+m}{m}.$$

Thus $\binom{d+m}{m} = \frac{(d+m)\dots(d+1)}{m!} = \frac{d^m}{m!} + \dots$ and

$$\mathrm{GKdim}(A) = \limsup_{d\to\infty} \log_d \binom{d+m}{m} = m.$$

Hence for any G-algebra A in n variables has $\mathrm{GKdim}_K(A) = n$.

Gel'fand-Kirillov dimension: examples and properties

Free associative algebra $T = K\langle x_1,\dots,x_n\rangle, n \geq 2$

$\dim V_d = n^d, \dim V^d = \frac{n^{d+1}-1}{n-1}$. Note, that $\frac{n^{d+1}-1}{n-1} > n^d$.
Since $\log_d n^d = d \log_d n = \frac{d}{\log_n d} \to \infty, d \to \infty$, it follows that $\mathrm{GKdim}(T) = \infty$.

Properties

- $\mathrm{GKdim}\, M = \sup\{\mathrm{GKdim}(N) : N \in A - mod,\ N \subseteq M\}$,
- $\mathrm{GKdim}\, A = \sup\{\mathrm{GKdim}(S) : S \subseteq A,\ S \text{ fin. gen. subalgebra}\}$

Hence, if $|K| = \infty$, then $\mathrm{GKdim}(K[[x_1,\dots,x_n]]) = \infty$ for $n \geq 1$.

Lemma (R is commutative)

(i) *Let R be a commutative affine K-algebra. Then (by Noether normalization) $\exists S = K[x_1,\dots,x_t] \subseteq R$ and R is finitely generated S-module. Then* $\mathrm{GKdim}_K R = \mathrm{Kr.dim}\, S = t$.

(ii) *If R is an integral domain,* $\mathrm{GKdim}_K R = \mathrm{tr.deg}_K \mathrm{Quot}(R)$.

For any K-algebra R: $\mathrm{GKdim}\, R[x_1,\dots,x_m] = \mathrm{GKdim}\, R + m$.
Curiosity: $\mathrm{GKdim}(R) \in \{0,1\} \cup [2,+\infty)$.

Exactness

Let R be an affine algebra with finite standard fin.-dim. filtration, such that $\mathrm{Gr}\, R$ is left Noetherian. Then GKdim is exact on short exact sequences of fin. gen. left R-modules. That is,

$$0 \to L \to M \to N \to 0 \ \Rightarrow\ \mathrm{GKdim}\, M = \sup\{\mathrm{GKdim}\, L, \mathrm{GKdim}\, N\}$$

Gel'fand-Kirillov dimension for modules

There is an algorithm by Gomez-Torrecillas et. al., which computes Gel'fand-Kirillov dimension for finitely presented modules over G-algebras over ground field K. It is implemented e. g. in SINGULAR:PLURAL.

GKDIM$_K(F)$

Let A be a G-algebra in variables $x_1,\dots,x_n$.

- Input: Left generating set $F = \{f_1,\dots,f_m\} \subset A^r$
- Output: $k \in \mathbb{N}$, $k = \mathrm{GKdim}(A^r/M)$, where $M = {}_A\langle F\rangle \subseteq A^r$.
- $G =$ LEFTGRÖBNERBASIS$(F) = \{g_1,\dots,g_t\}$;
- $L = \{\mathrm{lm}(g_i) = x^{\alpha_i} e_s \mid 1 \leq i \leq t\}$;
- $N = {}_{K[x_1,\dots,x_n]}\langle L\rangle$;
- **return** $\mathrm{Kr.dim}(K[x_1,\dots,x_n]^r/N)$;

Gel'fand-Kirillov dimension for modules: example

Recall Legendre's example:

$$\mathfrak{D} := K\langle n, s_n \mid s_n n = n s_n + s_n\rangle \otimes_K K\langle x, \partial_x \mid \partial_x x = x\partial_x + 1\rangle.$$

Then $\operatorname{GKdim}_K \mathfrak{D} = 4$.

The Gröbner basis of the ideal P is

$$(x^2-1)\partial_x^2 + 2x\partial_x - n(1+n),\ (n+2)s_n^2 - (2n+3)xs_n + n + 1,$$
$$(n+1)s_n\partial_x - (n+1)x\partial_x - (n+1)^2.$$

The leading monomials are $x^2\partial_x^2, ns_n^2, ns_n\partial_x$. Hence

$$\operatorname{GKdim}_K \mathfrak{D}/P = \operatorname{Kr.dim} K[n, s_n, x, \partial_x]/\langle x^2\partial_x^2, ns_n^2, ns_n\partial_x\rangle = 2.$$

Elimination and GK-dimension

Lemma (MR, KL)

Let $I \subset A$ be a left ideal and $S \subset A$ be a subalgebra. Then

- *$I \cap S = 0$ implies $\operatorname{GKdim} A/I \geq \operatorname{GKdim} S$,*
- *$\operatorname{GKdim} A/I < \operatorname{GKdim} S$ implies $I \cap S \neq 0$.*

Recall: Bernstein's inequality

Let A be the n-th Weyl algebra over K with char $K = 0 = \operatorname{GKdim} K$, then $\operatorname{GKdim}(A) = 2n$.

Let $0 \neq M$ be an A-module, then $\operatorname{GKdim}_K M \geq n$.

Elimination and GK-dimension

Let $f \in \mathcal{F}$, such that $\operatorname{Ann}_{\mathfrak{D}} f \cap K[x_1, \ldots, x_n] = 0$. Then $\operatorname{GKdim}_K \mathfrak{D}/\operatorname{Ann}_{\mathfrak{D}} f \geq n$.

Proposition (Existence of elimination via dimension)

Let $\mathfrak{D} = \bigotimes_{i=1}^n \mathfrak{D}_i$, $\mathfrak{D}_i = K\langle x_i, \mathfrak{o}_i \mid \ldots\rangle$. Moreover, let $I \subset \mathfrak{D}$ and $\operatorname{GKdim} \mathfrak{D}/I = m$. Then for any subalgebra $S \subset \mathfrak{D}$, such that $\operatorname{GKdim} S \geq m+1$ one has $I \cap S \neq 0$.

Application: For I such that $\operatorname{GKdim} \mathfrak{D}/I = m$ we guarantee that $2n - (m+1) = 2n - m - 1$ variables can be eliminated from I, for instance, if $m = n$, we can eliminate

- all but one operators,
- all but one coordinate variables.

More applications will follow ... in the parts, which follow.

Part III. Ore localization.

Localization in commutative case

Let A be a **commutative** Noetherian domain and

S a multiplicatively closed set in A, where $0 \notin S$.

The **localization** of A w.r.t S is a ring $A_S := S^{-1}A$ together with an injective homomorphism $\phi : A \to A_S$, such that

(i) for all $s \in S$ $\phi(s)$ is a unit in A_S,

(ii) for all $f \in A_S$, $\exists a \in A, s \in S$ s. t. $f = \phi(s)^{-1}\phi(a)$.

Ore Localization and its recognition
Ore localization on modules
The complete annihilator program
Localization: commutative case
Localization: non-commutative case, Ore
Smallest Ore localizations

Example

Let $A = K[x_1, \ldots, x_n]$.

- for $f \in A \setminus K$, consider $S = \{f^i : i \in \mathbb{N}\}$. Then

$$S^{-1}A \cong K[x_1, \ldots, x_n, \tfrac{1}{f}].$$

  this type is called a **monoidal localization**.

- Another instance: for $f_1, \ldots, f_m \in A \setminus K$, defining $S = \{f_1^{i_1} \cdot \ldots f_m^{i_m} : i_j \in \mathbb{N}\}$ results in

$$K[x_1, \ldots, x_n, \tfrac{1}{f_1}, \ldots, \tfrac{1}{f_m}] \cong K[x_1, \ldots, x_n, \tfrac{1}{f_1 \cdots f_m}]$$

$$\cong (\{(f_1 \cdots f_m)^i : i \in \mathbb{N}\})^{-1}K[x_1, \ldots, x_n].$$

VL Elements of CAAN
Ore Localization and its recognition
Ore localization on modules
The complete annihilator program
Localization: commutative case
Localization: non-commutative case, Ore
Smallest Ore localizations

Example

Let $A = K[x_1, \ldots, x_n]$.

- If $S = A^* := A \setminus \{0\}$, then $S^{-1}A \cong \mathrm{Quot}(A) = K(x_1, \ldots, x_n)$.
  this type is called a **rational localization**.
- For $p \in K^n$, consider $\mathfrak{m}_p := \langle x_1 - p_1, \ldots, x_n - p_n \rangle$, a maximal ideal in $K[x_1, \ldots, x_n]$. Define $S = K[x_1, \ldots, x_n] \setminus \mathfrak{m}_p$. Then

$$S^{-1}A = K[x_1, \ldots, x_n]_p = \{\frac{g}{h} \mid g \in K[x_1, \ldots, x_n],\ h \notin \mathfrak{m}_p\}$$

  this type is called a **geometric localization**, it is widely used in algebraic geometry.

VL Elements of CAAN
Ore Localization and its recognition
Ore localization on modules
The complete annihilator program
Localization: commutative case
Localization: non-commutative case, Ore
Smallest Ore localizations

Example

Let $A = K[[x_1, \ldots, x_n]]$.

- If $S = A^* := A \setminus \{0\}$, then $S^{-1}A \cong \mathrm{Quot}(A) = K((x_1, \ldots, x_n))$.
- Notably, $K[[x_1, \ldots, x_n]]$ is a local ring (i. e. there is exactly one maximal ideal). Thus for $f := x_1 \cdots x_n$ and $S = \{f^i : i \in \mathbb{N}\}$ one has $S^{-1}A \cong \mathrm{Quot}(A) = K((x_1, \ldots, x_n))$.

Ore localization

Let A be a **non-commutative** Noetherian domain and S a multiplicatively closed set in A, where $0 \notin S$.
If S is additionally an Ore set in A, then $\exists\ S^{-1}A$.

Ore condition

For all $s_1 \in S,\ r_1 \in A$ there exist $s_2 \in S,\ r_2 \in A$, such that

$$\mathbf{r_1} s_2 = \mathbf{s_1} r_2, \quad \text{that is} \quad s_1^{-1} r_1 = r_2 s_2^{-1}.$$

Ore condition holds $\Rightarrow$ S is an Ore set in A.

Ore Localization and its recognition
Ore localization on modules
The complete annihilator program
Localization: commutative case
Localization: non-commutative case, Ore
Smallest Ore localizations

Example

- Let $S = A^* := A \setminus \{0\}$. Then $S^{-1}A \cong \mathrm{Quot}(A)$ (quotient division ring of a domain).
- If $K \subsetneq S \subsetneq A^*$, then $A \to A_S \to \mathrm{Quot}(A)$,
- For any S, $S^{-1}A$ is an A-module (not finitely generated),
- in general A is not an $S^{-1}A$-module.

S^{-1} gives rise to a functor A-mod $\to S^{-1}A$-mod.

Smallest localizations

We take A to be one of model algebras and $f \in A \setminus K$. We will analyze, whether $S = \{f^i : i \in \mathbb{N}\}$ is an Ore set in A.

Weyl algebra: S is an Ore set

Suppose we are given $g = \sum_{j=0}^{d} b_j(x)\partial^j \in A_1$ with $b_d \neq 0$ and f^k for a fixed $k \in \mathbb{N}$.

For $j \in \mathbb{N}$ and $i+1 \geq j$ one has $\partial^j \cdot f^{i+1} = f^{i-j+1} \cdot (f^j \partial^j + v_{ij})$, where the terms of $v_{ij} \in A_1$ have degree at most $j-1$ and contain derivatives up to $f^{(j)}$. Then

$$\mathbf{g} \cdot f^{d+k} = \sum_{j=0}^{d} b_j(x)\partial^j \cdot f^{d+k} = \mathbf{f^k} \cdot \sum_{j=0}^{d} b_j(x)(f^j\partial^j + v_{j+k,j})f^{d-j}.$$

VL
Elements of CAAN

Smallest localizations: shift algebra

S is not Ore in $A = K\langle x, s \mid sx = (x+1)s\rangle$

Take s and $f^k(x) \in S$ and suppose, that $\exists f^{\ell}(x)$ and $\exists t \in A$, such that $sf^{\ell}(x) = f^k(x)t$. Thus $f^k(x)t = f(x+1)^{\ell}s$. But

$f(x) \nmid f(x+1)$ for $f \notin K$, thus such $t \in A$ does not exist.

Let us introduce the notion of **Ore closure** of a multiplicatively closed set S: $\mathcal{M}(S)$ is the smallest (w.r.t inclusion) two-sided multiplicative superset of S, which has an Ore property in A.

Smallest localizations: shift algebra

Lemma (L.–Schindelar, 2011)

For the shift algebra, $\mathcal{M}(S) = \{f^n(x \pm z) \mid n, z \in \mathbb{N}_0\}$.

Given $g = \sum_{j=0}^{d} b_j(x)s^j \in A$ with $b_d \neq 0$ and $h(x) = f^k(x + z_0)$ with $k \in \mathbb{N}, z_0 \in \mathbb{Z}$. Let us define $g_f(x) := \prod_{i=0}^{d} h(x-i) \in S$. Then

$$\mathbf{g} \cdot g_f(x) = \mathbf{h}(\mathbf{x})^{\mathbf{d}} \cdot \sum_{j=0}^{d} b_j(x)\Big(\prod_{i=0, i\neq j}^{d} h(x+j-i)s^j\Big).$$

Smallest localizations: shift algebra, other set

Consider $S = \{s^i : i \in \mathbb{N}\}$. Then S is an Ore set in A.

This follows from

$$\sum_{\mathbf{i}} \mathbf{a_i(x)s^i} \cdot s^k = \mathbf{s^k} \cdot \sum_i a_i(x-k)s^i$$

The resulting algebra is already mentioned **full shift algebra**:

$$(\{s^i : i \in \mathbb{N}\})^{-1} K\langle x, s \mid sx = (x+1)s\rangle \cong$$
$$K\langle x, s, s^{-1} \mid sx = (x+1)s,\ s^{-1}x = (x-1)s^{-1}, s^{-1}s = 1\rangle$$

Smallest localizations: quantum plane

$S = \{f^i : i \in \mathbb{N}\}$ is not Ore in $A = K(q)\langle x, y \mid yx = qxy\rangle$.

Lemma (L.–Schindelar, 2011)

$\mathcal{M}(S) = \{f^n(q^{\pm z}x) \mid n, z \in \mathbb{N}_0\}$ *is an Ore set in A.*

For any $g(x) \in K[x]$ one has $y^m g(x) = g(q^m x)y$.
Suppose we are given $g = \sum_{j=0}^{d} b_j(x)y^j \in A$ and $h(x) = f^k(q^\ell x) \in S_1$. Let us define $g_f(x) := \prod_{i=0}^{d} h(q^{-i}x) \in S$.
Then

$$\mathbf{g} \cdot g_f(x) = \sum_{j=0}^{d} b_j(x)y^j \cdot \prod_{i=0}^{d} h(q^{-i}x) = \mathbf{h}(\mathbf{x}) \cdot \sum_{j=0}^{d} b_j(x) \prod_{i=0, i\neq j}^{d} h(q^{j-i}x)y^j.$$

With Ore localization we can recognize, that

$$K(X)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_m; \sigma_m, \delta_m] \cong (K[X]\setminus\{0\})^{-1} K\langle X, \partial_1, \ldots, \partial_m \mid \ldots\rangle$$

and the functor S^{-1} connects categories of modules.

Algorithmic aspects

Algorithmic computations over $S^{-1}A$ can be replaced **completely** with computations over A.
Keywords: **integer strategy, fraction-free strategy**.
For instance, a Gröbner basis theory over A induces a Gröbner basis theory over $S^{-1}A$.

There are implementations for the rational localization $K(X)\langle\partial_1, \ldots\rangle$.

Induced Gröbner basis theory in the localization

Let A be a G-algebra $K[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \mid \ldots \rangle$.
Suppose that $S = K[x_1, \ldots, x_n] \setminus \{0\}$ is an Ore set in A.
Moreover, let $\prec_X$ be an admissible monomial ordering on A, having the elimination property for x, that is

$$1 \prec_X x^\alpha \prec_X \partial_i, \ \forall \alpha \in \mathbb{N}^n, \ \forall 1 \leq i \leq n.$$

Lemma

A Gröbner basis of a submodule $N \subset A^r$ w.r.t $\prec_X$ is a non-reduced Gröbner basis of a submodule

$$S^{-1}N \subset (S^{-1}A)^r = K(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \mid \ldots \rangle^r.$$

Properties of localized modules

Let A be a K-algebra and $S \subset A$ a mult. closed Ore set in A. Moreover, let

- $M \cong A^n / A^m P$, a finitely presented left A-module,
- $\mathcal{F}$ a left A-module,
- $\widetilde{\mathcal{F}}$ a left $S^{-1}A$-module.

- $S^{-1}M \ \cong \ (S^{-1}A)^n / (S^{-1}A)^m P$.
- $\mathrm{Sol}_A(M, \widetilde{\mathcal{F}}) \cong \mathrm{Sol}_{S^{-1}A}(S^{-1}M, \widetilde{\mathcal{F}})$,
- Assume, that $\widetilde{\mathcal{F}} \subset \mathcal{F}$ as left A-modules. Then

$$\mathrm{Sol}_A(M, \widetilde{\mathcal{F}}) \subseteq \mathrm{Sol}_A(M, \mathcal{F}),$$

Properties of localized modules

Message: In order to compute generalized solutions, work over unlocalized ring and thus employ target spaces, having torsion under localization.

Technology: the information, obtained for the localized module (and homomorphism of such etc.), can be and should be used for studying the original module (and homomorphism of such etc.).

Properties of localized modules

Here is a typical situation of behaviour of modules under localization. Let M_i be A-modules, satisfying

$$0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_i \subsetneq \ldots \subsetneq M_j \subsetneq \ldots \subsetneq M_k \subsetneq \ldots \subsetneq M_r \subsetneq A$$

After applying S^{-1} to this sequence, we obtain

$$0 = \ldots = S^{-1}M_{i-1} \subsetneq S^{-1}M_i = \ldots = S^{-1}M_{j-1} \subsetneq S^{-1}M_j \subset \ldots$$

$$\ldots \subset S^{-1}M_k \subsetneq S^{-1}M_{k+1} = \ldots = S^{-1}M_r = A.$$

Elimination, dimension and localization

Lower bound for nontrivially localizable modules

Suppose that $I, S \subset \mathfrak{D}$ are such that

- S is an Ore set in $\mathfrak{D}$ (so $S^{-1}\mathfrak{D}$ exists)
- $(S^{-1}\mathfrak{D})I \neq S^{-1}\mathfrak{D}$ (i. e. I is proper in the localized algebra).

Then $I \cap S = 0$, what implies $\text{GKdim}\, \mathfrak{D}/I \geq \text{GKdim}\, KS$, where KS is the monoid algebra.

Note, that for every $J \in S^{-1}\mathfrak{D}$ there exists $I \in \mathfrak{D}$ such that $S^{-1}\mathfrak{D}I = S^{-1}\mathfrak{D}J$ (idea: clear denominators).

In general, if $S^{-1}\mathfrak{D}L \neq S^{-1}\mathfrak{D}$, one has

$$\text{GKdim}\, S^{-1}\mathfrak{D}/(S^{-1}\mathfrak{D})L \geq \text{GKdim}\, \mathfrak{D}/L.$$

GK-dimension and localization

Drawback of Gel'fand-Kirillov dimension of localized algebras: it is mathematically hard to determine. It is known, that $\text{GKdim}\, S^{-1}A \geq \text{GKdim}\, A$.

Lemma (Very exceptional result)

Let A be the n-th Weyl algebra, $S = K[x_1, \ldots, x_n] \setminus \{0\}$. Then $\text{GKdim}\, S^{-1}A = \text{GKdim}\, A = 2n$.

In the analogous situation for A being n-th shift, q-Weyl algebra or a quantum space, we have $\text{GKdim}\, S^{-1}A \geq 3n$.

Lemma (Corollary from Makar-Limanov)

Let $K = \mathbb{C}$, A be the n-th shift algebra and $S = K[x_1, \ldots, x_n] \setminus \{0\}$. Then $\text{GKdim}\, S^{-1}A = 3n > 2n = \text{GKdim}\, A$.

The complete annihilator program

Let $\mathcal{G} \subset \mathcal{F}$ be function spaces, i. e. K-vector spaces and left $\mathfrak{D}$-modules over a fixed operator algebra $\mathfrak{D}$.

Let $f \in \mathcal{F}$, then $\text{Ann}^{\mathcal{F}}_{\mathfrak{D}}\, f := \{p \in \mathfrak{D} : pf = 0 \in \mathcal{F}\}$ is the **annihilator** of f, which is a left ideal in $\mathfrak{D}$.

Let $I \subsetneq \mathfrak{D}$ be an ideal and suppose, that $\dim_K(\mathcal{G}) < \infty$.
I is called **the complete annihilator of $\mathcal{G}$ over $\mathfrak{D}$**, if the following properties hold:
"most powerful": if $\forall g \in \mathcal{G}$ $rg = 0$ for $r \in \mathfrak{D}$, then $r \in I$
"unfalsified": $\text{Sol}_{\mathfrak{D}}(\mathfrak{D}/I,\ \mathcal{F}) = \mathcal{G}$.

The complete annihilator program

There exists no general algorithm, which can compute the complete annihilator program of f over $\mathfrak{D}$ (where $\mathfrak{D}$ is an algebra with polynomial coefficients).

Therefore one investigates some classes of f and develops special methods for the classes.

One of successes is **computational D-module theory**, where among other one can compute the complete annihilators of

$$f(\mathbf{x}, \mathbf{s}) = f_1(x_1, \ldots, x_n)^{s_1} \cdot \ldots \cdot f_m(x_1, \ldots, x_m)^{s_m},\ f_i(\mathbf{x}) \in K[x_1, \ldots, x_n]$$

$$\text{over}\ \ \mathfrak{D} = \bigotimes_{i=1}^{n}{}_K K\langle x_i, \partial_i \mid \partial_i x_i = x_i \partial_i + 1\rangle \otimes_K K[s_1, \ldots, s_m]$$

in an algorithmic way. There are implementations.

Some computational D-module theory

Let char $K = 0$,
$D_n(K) = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \mid \partial_j x_i = x_i \partial_j + \delta_{ij} \rangle$ be the n-th Weyl algebra and $D_n[s] = D_n \otimes_K K[s]$.

Theorem (J. Bernstein, 1971/72)

Let $f(x) \in \mathbb{C}[x_1, \dots, x_n]$. Then there exist

- *an operator $P(s) \in D_n \otimes_{\mathbb{C}} \mathbb{C}[s]$,*
- *a monic polynomial $0 \neq b_f(s) \in \mathbb{C}[s]$ of the smallest degree (called* **the global Bernstein-Sato polynomial***),*

such that for arbitrary s the following functional equation holds

$$P(s) \bullet f^{s+1} = b_f(s) \cdot f^s.$$

Let $\mathrm{Ann}_{D[s]}(f^s) = \{Q(s) \in D[s] \mid Q(s) \bullet f^s = 0\} \subset D[s]$ be the annihilator, then $P(s)f - b_f(s) \in \mathrm{Ann}_{D[s]}(f^s)$ holds.

Some computational D-module theory

Some very easy examples:

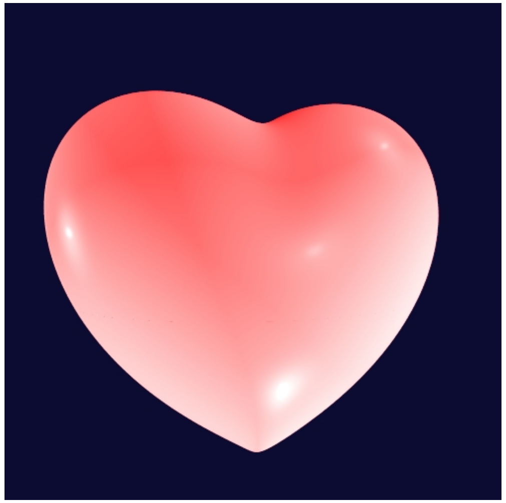
$$\begin{aligned} \partial_x \bullet (x)^{s+1} &= (s+1) \cdot (x)^s, \\ (1/4)\partial_x^2 \bullet (x^2)^{s+1} &= (s+1)(s+1/2) \cdot (x^2)^s, \\ (2x\partial_x + \partial_x - 4s - 4) \bullet (x^2+x)^{s+1} &= (s+1) \cdot (x^2+x)^s. \end{aligned}$$

Some facts

- M. Kashiwara: all roots of $b_f(s)$ are negative rationals
- -1 is always a root; in general the roots lie in $(-n, 0)$
- $b_f(s) = s+1$ if and only if $V(f)$ is smooth
- B. Malgrange: if $b_{f,p}(\xi) = 0$ (local Bernstein-Sato polynomial at $p \in V(f)$), then $e^{2i\pi\xi}$ is an eigenvalue of the action on monodromy
- Complicated Bernstein-Sato polynomials appear for such f, that $V(f)$ possess complicated singularities

More interesting D-module example

Consider $f = (x^2 + \frac{9}{4}y^2 + z^2 - 1)^3 - x^2 z^3 - \frac{9}{80} y^2 z^3 \in K[x, y, z]$.



Numerology of Bernstein data

Then $\mathrm{Ann}_{\mathbf{D[s]}}\, \mathbf{f^s}$ has 13 generators with leading terms

$4617 \cdot y^3 \partial_x,\ 513 \cdot x^2 y \partial_x, \dots, 102400 \cdot x^2 z^5 \partial_y,\ 37428480 \cdot y^4 z^5 \partial_y;$

Bernstein-Sato poly: $b_f(s) = (s+1)^2 \cdot (s + \frac{2}{3}) \cdot (s + \frac{4}{3}) \cdot (s + \frac{5}{3})$

A reduced operator $\mathbf{P_f(S)}$ has 1261 terms, here some leading part of them

$$\left(\frac{1}{24} x y^2 z^3 - \frac{1}{5760} x y^2 z^2\right) \partial_x^3 \partial_z^2 + \left(\frac{7084781}{177292800} y z^3 - \frac{1}{4104} y z^2\right) \partial_x^2 \partial_y \partial_z^2$$

Bernstein-Sato and singularities

- $\mathrm{Sing}(f) = V_1 \cup V_2$, where
- $V_1 = V(\langle x^2 + 9/4y^2 - 1, z)$ an ellipse at $z = 0$ plane;
- $V_2 = V(\langle x, y, z^2 - 1\rangle)$ consists of 2 different points;
- $V_3 = V(\langle 19x^2 + 1, 171y^2 - 80, z\rangle)$ consists of 4 different points; moreover, $V_3 \subset V_1$, $V_2 \cap V_3 = \emptyset$.

L. and Martín-Morales, 2012: algorithm for constructing a stratification of $\mathbb{C}^3$ into constructible sets such that $b_{f,p}(s)$ is constant on each stratum.

$$b_{f,p}(s) = \begin{cases} 1 & p \in \mathbb{C}^3 \setminus V(f), \\ s+1 & p \in V(f) \setminus (V_1 \cup V_2), \\ (s+1)^2(s+4/3)(s+2/3) & p \in V_1 \setminus V_3, \\ (s+1)^2(s+4/3)(s+5/3)(s+2/3) & p \in V_3, \\ (s+1)(s+4/3)(s+5/3) & p \in V_2. \end{cases}$$

Weyl closure

Another success of computational D-module theory is the possibility to compute the **Weyl closure** of certain ideals.

Let A be a K-algebra, $S \subset A$ a m. c. Ore set in A. Moreover, let $0 \neq J \subsetneq S^{-1}A$ a left ideal. The **restriction** of J to A is the ideal $(S^{-1}A)J \cap A$.

Let A be the n-th Weyl algebra, $S = K(x_1, \dots, x_n) \setminus \{0\}$ and a left ideal J satisfies $\dim_{K(x_1,\dots,x_n)} S^{-1}A/(S^{-1}A)J < \infty$. Then the Weyl closure of J is defined to be the restriction of J to A and there is an algorithm to compute it in finitely many steps.

There are implementations of algorithms, computing Weyl closure after H. Tsai (Macaulay2, recently in Singular:Plural).

Weyl closure

Example (1st Weyl algebra)

Let $I = \langle (x^3+2)\partial_x - 3x^2 \rangle \subset D_1$. Gröbner basis of J is then

$$\{\partial_x^3 + x\partial_x - 3,\ x\partial_x^2 - 2\partial_x,\ x^2\partial_x + \partial_x^2 - 3x\}$$

w.r.t degree reverse lexicographical ordering and

$$\{(x^3+2)\partial_x - 3x^2,\ {\partial_x}^2 + x^2\partial_x - 3x\}$$

w.r.t the ordering, eliminating x (compatible with localization).

Open problem

Can one develop algorithms for computing analogous closure for other model algebras? Possible bottleneck: localizations of other algebras are more involved, as we know from before.

Bernstein-Sato polynomial for varieties

In the following, $f^s := f_1^{s_1} \cdots f_r^{s_r}$.

Theorem (Budur, Mustaţă and Saito, 2006)

Let char $K = 0$. *For every r-tuple $f = (f_1, \dots, f_r) \in K[\mathbf{x}]^r$ there exists a non-zero univariate polynomial $b(\xi) \in K[\xi]$ and r differential operators $P_1(S), \dots, P_r(S) \in D_n\langle S\rangle$ such that*

$$\sum_{k=1}^{r} P_k(S) f_k \bullet f^s = b(s_{11} + \dots + s_{rr}) \cdot f^s.$$

Here D_n is the n-th Weyl algebra in x_i, ∂_i and

$$D_n\langle S\rangle = D_n \otimes_K K\langle s_{11}, \dots, s_{nn} \mid \forall\ 1 \leq i,j,k,l \leq n$$

$$s_{ij}s_{kl} - s_{kl}s_{ij} = \delta_{jk}s_{il} - \delta_{il}s_{kj}\rangle$$

Bernstein-Sato polynomial for varieties

Theorem (Andres–L.–Martín-Morales, ISSAC 2009)

Let $f = (f_1, \ldots, f_r)$ be an r-tuple in $K[x]^r$ and let $D_n\langle \partial_t, S\rangle$ be the K-algebra generated by ∂_t and S over D_n subject to relations on S and $\{s_{ij} \cdot \partial t_k - \partial t_k \cdot s_{ij} = \delta_{jk} \partial t_i\}$. Consider the left ideal in $D_n\langle \partial_t, S\rangle$

$$F := \left\langle s_{ij} + \partial t_i f_j \,,\; \partial x_m + \sum_{k=1}^{r} \frac{\partial f_k}{\partial x_m} \partial t_k \;\middle|\; \begin{array}{c} 1 \le i,j \le r \\ 1 \le m \le n \end{array} \right\rangle.$$

Then $\mathrm{Ann}_{D\langle S\rangle}(f^s) = D_n\langle \partial_t, S\rangle F \cap D_n\langle S\rangle$.

Thus, this is another type of objects, for which complete annihilator program is successful.

Part IV. Purity.

Dimension function

Let A be a Noetherian algebra. A dimension function δ assigns a value $\delta(M)$ to each finitely generated A-module M and satisfies the following properties:

(i) $\delta(0) = -\infty$.

(ii) If $0 \to M' \to M \to M'' \to 0$ is exact sequence, then $\delta(M) \ge \sup\{\delta(M'), \delta(M'')\}$ with equality if the sequence is split.

(iii) If P is a (two-sided) prime ideal with $P \subseteq \mathrm{Ann}_A(M)$ and M is a torsion module over A/P, then $\delta(M) \le \delta(A/P) - 1$.

- generalized Krull dimension is an exact dimension function
- Gel'fand-Kirillov dimension is a dimension function, not always exact

Purity w.r.t dimension function

Let A be a K-algebra and δ a dimension function on A-mod. A module $M \neq 0$ is δ-**pure** (or δ-homogeneous), if

$$\forall 0 \neq N \subseteq M, \quad \delta(N) = \delta(M).$$

- A simple module is pure. Thus, purity is a useful weakening of the concept of simplicity of a module.
- Unlike simplicity, the purity (w.r.t a dimension function) is algorithmically decidable over many common algebras.

M. Barakat, A. Quadrat: Algorithms for the computation of the purity filtration of a module with $\delta =$ homological grade; there are several implementations: in HOMALG, OREMODULES(MAPLE) and SINGULAR:PLURAL.

Purity with respect to a dimension function

Lemma (L.)

Let A be a K-algebra and δ a dimension function on A-mod. Moreover, let $0 \neq M_1, M_2 \subset N$ be two δ-pure modules with $\delta(M_1) = \delta(M_2)$. Then

the set of δ-pure submodules (of the same dimension) of a module is a lattice, i. e.

1. *$M_1 \cap M_2$ is either 0 or it is δ-pure with $\delta(M_1 \cap M_2) = \delta(M_1)$,*
2. *$M_1 + M_2$ is δ-pure with $\delta(M_1 + M_2) = \delta(M_1)$.*

Ubiquity of pure modules

Consider purity with respect to Gel'fand-Kirillov dimension.

Lemma (L.)

Let A be a G-algebra, $S \subset A$ a m. c. Ore set in A. Let $\mathcal{M}$ be a set of left A-modules M, satisfying $S^{-1}M \neq 0$ and having dimension GKdim KS, *where KS is the monoid algebra. Then $\mathcal{M}$ consists of pure modules.*

Example (Pure modules)

- modules of Krull dimension 0 over $K[x_1, \ldots, x_n]$, i. e. modules M, such that $\dim_K M < \infty$
- any set of modules of smallest possible dimension in A, for instance holonomic modules over the n-th Weyl algebra over a field with char $K = 0$; it is known that they have GK dimension n over K.

Ubiquity of pure modules

Recall

Let A be an operator algebra over $K[x_1, \ldots, x_n]$ and $S = K[x_1, \ldots, x_n] \setminus \{0\} \subset A$ be a m. c. Ore set in A. A left A-module M is called D-**finite**, if $\dim_{K(x_1,\ldots,x_n)} S^{-1}M < \infty$.

Thus D-finite modules are pure.

Note: we can do much more with the concept of purity

We can consider pure modules of any reasonable dimension, without restricting ourselves to the modules of smallest possible dimension!

Pure functions and operations with them

Let $\mathfrak{D}$ be an operator algebra and $\mathcal{F}$ an $\mathfrak{D}$-module. A torsion element $f \in \mathcal{F}$ (that is a "function" having nonzero annihilator) is called **pure**, is the corresponding left $\mathfrak{D}$-module $\mathfrak{D}f \cong \mathfrak{D}/\operatorname{Ann}_{\mathfrak{D}} f$ is pure.

This definition generalizes both the notion of Zeilberger-*holonomic* or *D-finite* function as well as some other.

Lemma (L.)

Let $f \in \mathcal{F}$ be a pure function. Then for any $\mathfrak{o} \in \mathfrak{D} \setminus \{0\}$ $\quad h = \mathfrak{o} f$ is pure as well.

Proof: $\mathfrak{D}g = \mathfrak{D}\mathfrak{o}f \subset \mathfrak{D}f$ is a natural submodule, hence it is pure. Moreover, $\operatorname{Ann}_{\mathfrak{D}} \mathfrak{o}f =$

$$\{r \in \mathfrak{D} : r(\mathfrak{o}f) = (r\mathfrak{o})f = 0\} = \{s \in \operatorname{Ann}_{\mathfrak{D}} f : \exists r \in \mathfrak{D}, s = r\mathfrak{o}\} =$$

$$\operatorname{Ann}_{\mathfrak{D}} f : \mathfrak{o} = \operatorname{Ker}_{\mathfrak{D}}(\mathfrak{D} \to \mathfrak{D}/\operatorname{Ann}_{\mathfrak{D}} f, \ 1 \mapsto \mathfrak{o}) \text{ is computable.}$$

Operations with pure functions

Lemma (L.)

Let $f, g \in \mathcal{F}$ be pure functions. Then for any $\mathfrak{p}, \mathfrak{q} \in \mathfrak{D} \setminus \{0\}$ $h = \mathfrak{p}f + \mathfrak{q}g$ is pure as well.

Proof: by the previous lemma $M_f = \mathfrak{D}\mathfrak{p}f$ and $M_g = \mathfrak{D}\mathfrak{q}g$ are pure modules. By another lemma before $M_f + M_g$ is pure. Hence $\mathfrak{D}h \subseteq M_f + M_g$ is pure as well.
Moreover, $(\mathrm{Ann}_{\mathfrak{D}} f : \mathfrak{p}) \cap (\mathrm{Ann}_{\mathfrak{D}} g : \mathfrak{q}) \subseteq \mathrm{Ann}_{\mathfrak{D}} h$.

More operations, preserving the purity, are under investigation.

Observation : many (but not all) special functions give rise to pure modules.

Identities, Elimination, Purity Filtration

Let $0 \to M_1 \to M_2 \to M_2/M_1 \to 0$ be an exact sequence of fin. pres. $\mathfrak{D}$-modules. Moreover, let $\mathcal{F}$ be an arbitrary $\mathfrak{D}$-module. Then we have that $\mathrm{Sol}_{\mathfrak{D}}(M_2/M_1, \mathcal{F}) \subseteq \mathrm{Sol}_{\mathfrak{D}}(M_2, \mathcal{F})$.

If $\mathcal{F}$ is injective $\mathfrak{D}$-module, the natural map $\mathrm{Sol}_{\mathfrak{D}}(M_2, \mathcal{F}) \to \mathrm{Sol}_{\mathfrak{D}}(M_1, \mathcal{F})$ is surjective (not true for general $\mathcal{F}$).

Purity filtration with $\delta = \mathrm{GKdim}$

Let $\mathfrak{D}$ be a Noetherian domain, being Auslander-regular and Cohen-Macaulay algebra with $\mathrm{GKdim}\, \mathfrak{D} = n$.
Given a fin. pres. $\mathfrak{D}$-module M of dimension $n > d \geq 0$, then the purity filtration of M is the sequence

$$M = M_{n-d} \supset M_{n-d+1} \ldots \supset M_{n-1} \supset M_n = 0.$$

where $\mathrm{GKdim}\, M_{n-(d-i)} = d - i$. Moreover, $M_{n-d+k}/M_{n-d+k+1}$ is either 0 or pure of dimension $d - k$.

Identities, Elimination, Purity Filtration

Consider the mixed system, annihilating Legendre polynomials

$$\mathfrak{D} = K\langle n, s_n \mid s_n n = n s_n + s_n\rangle \otimes_K K\langle x, \partial_x \mid \partial_x x = x\partial_x + 1\rangle.$$

$$M = \mathfrak{D}/P,$$

$$P = \langle (x^2-1)\partial_x^2 + 2x\partial_x - n(1+n),\ (n+2)s_n^2 - (2n+3)x s_n + n + 1,$$
$$(n+1)(s_n\partial_x - x\partial_x + n + 1)\rangle.$$

$$\mathrm{GKdim}\, \mathfrak{D} = 4, \quad \mathrm{GKdim}\, M = 2, \quad t(M) = M = \mathfrak{D}/P.$$

The purity filtration of $M = t(M)$ is $0 \subsetneq M_3 \subsetneq M_2 = M$,

$$M_3 \cong \mathfrak{D}/\langle n+1, s_n, \partial_x\rangle \text{ with } \mathrm{GKdim}\, M_3 = 1.$$

What are the most general solutions $g(n, x)$ of this system?

Since $\partial_x(g) = 0$, one has $g(n, x) = g(n)$.
however, $g(n)$ should not be identically zero:
in case $n \in \{-1, 0, 1, \ldots\}$, one can select $g(-1) \in K$ arbitrary (step of the jump function).

Localization

The ideal $\langle n+1, s_n\rangle$ is two-sided and maximal. Hence the submodule M_3 vanishes under any nontrivial Ore localization w. r. t $S \subset K\langle n, s_n \ldots\rangle$, for instance when $n \in S$ or $s_n \in S$ (then s_n^{-1} is present and therefore $n \in \mathbb{Z}$ should hold). And $S^{-1}M$ is then a pure module.

The purity filtration of $M = t(M)$ is $0 \subsetneq M_3 \subsetneq M_2 = M$.
The pure part of GK dimension 2 is $t(M)/M_3 \cong$

$$\mathfrak{D}/\langle (x^2-1)\partial_x^2 + 2x\partial_x - n(1+n),\ (n+2)S_n^2 - (2n+3)xS_n + n + 1,$$

$$(1-x^2)\partial_x + (n+1)S_n - (n+1)x\rangle.$$

For further investigations of M over localizations w.r.t. n or S_n one should then take the simplified equations from the ideal P' above.

Elimination leads to new identities

The elimination property guarantees, that 1 arbitrary variable of $\mathfrak{D}$ can be eliminated from P and from P'; so one gets for instance

$$\mathbf{x\text{-}free}:\ (n+1)(n+2)\cdot\left((S_n^2-1)\partial_x - (2n+3)S_n\right) \bullet P_n(x) = 0,$$

$$\mathbf{n\text{-}free}:\ (1-x^2)\cdot\left((S_n^2 - 2xS_n + 1)\partial_x - S_n\right) \bullet P_n(x) = 0.$$

The hypergeometric series is defined for $|z| < 1$ and $-c \notin \mathbb{N}_0$ as follows:

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

We derive two annihilating ideals from the anihilator of ${}_2F_1(a,b,c;z)$:

- J_a which does not contain a,
- J_c which does not contain c,

and analyze corresponding modules for purity.

Case J_a

The ideal in $\mathfrak{D} = K[b,c,z]\langle Sb, Sc, Dz \mid \ldots\rangle$ is generated by:

$$bcSb - czDz - bc$$
$$bSbSc - bSc + cSc - c$$
$$bSb^2 - zSbDz - bSb + Sb^2 - Sb$$
$$b^2Sb - bzDz - b^2 + bSb - zDz - b$$
$$bzSbDz - z^2Dz^2 - bzDz - bSbDz + zDz^2 - bSb + bDz + b + Dz$$

Let $M = M_a = \mathfrak{D}/J_a$. Then $\mathrm{GKdim}\,\mathfrak{D} = 6, \mathrm{GKdim}\,M = 4$.

The purity filtration of $M = t(M)$

$0 \subsetneq M_5 = M_4 \subsetneq M_3 = M_2 = M$, where

$$M/M_5 \cong \mathfrak{D}/\langle bSb - zDz - b, zDzSc + cSc - c\rangle,\ \mathrm{GKdim}\, M/M_5 = 4$$

The purity filtration of $M = t(M)$

... and

$$M_5 \cong \mathfrak{D}/\langle c, Sb, b+1, zDz - Dz - 1\rangle,\ \mathrm{GKdim}\, M_5 = 2.$$

The solutions can be read off:

$$\delta_{c,0}\cdot\delta_{b,-1}\cdot(\ln(z-1)+k_0),\ k_0 \in K$$

Case J_c

The ideal in $\mathfrak{D} = K[b,c,z]\langle Sb, Sc, Dz \mid \ldots\rangle$ is generated by:

$$\begin{aligned} aSa - bSb - a + b \\ bSb^2 - SbzDz - bSb + Sb^2 - Sb \\ b^2Sb - bzDz - b^2 + bSb - zDz - b \\ abSb - azDz - ab + bSb - zDz - b \\ bSbzDz - z^2Dz^2 - bSbDz - bzDz + zDz^2 - bSb + bDz + b + Dz \end{aligned}$$

Let $M = M_c = \mathfrak{D}/J_c$. Then $\operatorname{GKdim} \mathfrak{D} = 6, \operatorname{GKdim} M = 4$.

The purity filtration of $M = t(M)$

$0 \subsetneq M_6 = M_5 = M_4 \subsetneq M_3 = M_2 = M$, where

$$M/M_6 \cong \mathfrak{D}/\langle bSb - zDz - b, aSa - zDz - a\rangle,\ \operatorname{GKdim} M/M_6 = 4.$$

The purity filtration of $M = t(M)$

... and

$$M_6 \cong \mathfrak{D}/\langle Sb, b+1, Sa, a+1, zDz - Dz - 1\rangle,\ \operatorname{GKdim} M_6 = 2.$$

The solutions:

$$\delta_{a,-1} \cdot \delta_{b,-1} \cdot (\ln(z-1) + k_0),\ k_0 \in K$$

Part V. Jacobson normal form.

One of the most important questions in algebra is undecidable in general:

Let A be a (Noetherian) K-algebra and M, N are two finitely presented A-modules. Can we decide, whether $M \cong N$ as A-modules?

Yet another application of localization as a functor:

Let $S \subset A$ be a m. c. Ore set, then $S^{-1}A$ exists.
Given an A-module homomorphism $\varphi : M \to N$ (M, N are finitely presented). Then there is an induced homomorphism of $S^{-1}A$-modules $S^{-1}\varphi : S^{-1}M \to S^{-1}N$.

Application to the isomorphism problem

If there exists such m. c. Ore set $\tilde{S} \subset A$, that $\tilde{S}^{-1}\varphi$ is not an isomorphism, then φ is not an isomorphism.

Invariants

Above we have seen several dimensions of modules, some of them are computable. What can one achieve with the help of localization?

- Let $S = A \setminus \{0\}$. Then the **rank** of f. g. A-module M is defined to be $\dim_{S^{-1}A} S^{-1}M$.
- Let $R = A[\partial; \sigma, \delta]$ for an integral domain A and $S = A \setminus \{0\}$. Then $S^{-1}M$ is a vector space over $\mathrm{Quot}(A) = S^{-1}A$ and $\dim_{S^{-1}R} S^{-1}M$ is an invariant of the module.

Jacobson, Teichmüller, Cohn

Let R be a non-commutative Euclidean domain and $M \in R^{m\times n}$. Then there exist

- unimodular matrices $U \in R^{m\times m}$, $V \in R^{n\times n}$;
- a matrix $D \in R^{m\times n}$ with elements $d_1, \dots, d_r$ on the main diagonal and 0 outside of the main diagonal ...
- such that $d_i || d_{i+1}$ (total divisibility), meaning $_{\mathfrak{D}}\langle d_{i+1}\rangle_{\mathfrak{D}} \subseteq {_{\mathfrak{D}}\langle d_i\rangle} \cap \langle d_i\rangle_{\mathfrak{D}}$

such that $U \cdot M \cdot V = D$.

In particular there is an isomorphism of R-modules

$$R^{1\times n}/R^{1\times m}M \cong R^{1\times n}/R^{1\times m}D.$$

Recognizing the localization

L.–Schindelar (2011, 2012) presented two algorithms, computing matrices U, V, D by using Gröbner bases.

A fraction-free algorithm performs only operations over polynomial (i.e. unlocalized) algebra. A minor modification allows to produce matrices U, V, D with polynomial entries.

Theorem (L.–Schindelar)

Let A be a G-algebra in variables $x_1, \dots, x_n, \partial$ and assume that $\{x_1, \dots, x_n\}$ generate a G-algebra $B \subsetneq A$. Suppose, there exists an admissible monomial ordering $\prec$ on A, satisfying $x_k \prec \partial$ for all $1 \le k \le n$. Then the following holds

- *B^* is multiplicatively closed Ore set in A.*
- *$(B^*)^{-1}A$ can be presented as an Ore extension of $\mathrm{Quot}(B)$ by the variable ∂.*

Example

Let A_1 be the polynomial and $B_1 = (K[x] \setminus \{0\})^{-1}A_1$ the rational Weyl algebra. Consider the matrix

$$M = \begin{bmatrix} \partial^2 - 1 & \partial + 1 \\ \partial^2 + 1 & \partial - x \end{bmatrix}.$$

The algorithm returns

$$D = \begin{bmatrix} x^2\partial^2 + 2x\partial^2 + \partial^2 - 2x\partial - 2\partial - x^2 - 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} -x\partial - \partial + x^2 + x + 1 & x\partial + \partial + x \\ \partial - x & -\partial - 1 \end{bmatrix},$$

$$V = \begin{bmatrix} 1 & 0 \\ x\partial^2 + \partial^2 + 2\partial - x + 1 & 1 \end{bmatrix}.$$

Unimodularity of Matrices

Let us analyze, under which localizations U, V will be invertible.

Indeed, V is unimodular over A_1, since it admits an inverse:

$$V^{-1} = \begin{bmatrix} 1 & 0 \\ -(x+1)\partial^2 + x - 2\partial - 1 & 1 \end{bmatrix}$$

On the contrary, U is NOT unimodular over A_1, since $U \cdot Z = W$ and W is first invertible in the localization:

$$Z = \begin{bmatrix} 2\partial + 2 & (x+1)\partial + x - 2 \\ 2(\partial - x) & (x+1)\partial - x^2 - x - 3 \end{bmatrix}, W = \begin{bmatrix} 0 & -4x^2 - 8x - 4 \\ 2 & 5x + 5 \end{bmatrix}$$

For the invertibility of W we need only to divide by $x + 1 =: f$.

Lifting the isomorphism

Let $f = x + 1$. Then U from above will be unimodular over any localization, where f is invertible. In particular, the smallest one, as we know, is $C_1 := S_f^{-1}A_1$, where $S_f = \{f^i : i \in \mathbb{N}\}$.

Thus the isomorphism of B_1-modules, provided by the Jacobson form, holds not only over $B_1 = (K[x] \setminus \{0\})^{-1}A_1$, but also over C_1.

General strategy: depending on the concrete questions, analyze U resp. V for unimodularity over localizations, less greedy than the rational one.

Note: the steps of such an analysis are algorithmic.

Recognize and lift localized problems

Strategical remarks for conclusion.

- use the information from the localized situation - for instance, implementations of numerous good algorithms - for the analysis of the unlocalized, "global" situation;
- in algorithms:

  perform fraction-free computations, if possible

  or keep track of operations, requiring localized computations
- use this tracking information and determine a smaller localization, where desired properties still hold. Lift the obtained results to that smaller localization.
- study obstructions to the lifting: this provides several cases, which again hints at the treatment of the problem at a global level by using local ones.
- obtain new powerful and useful results!

Merci beaucoup

pour votre attention!

http://www.singular.uni-kl.de/