simpcomp – A GAP toolbox for simplicial complexes

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May 26, 2010

1 The GAP package simpcomp

simpcomp [4] is an extension (a so called *package*) to GAP [6], the well known system for computational discrete algebra. The package enables the user to compute numerous properties of (abstract) simplicial complexes, provides functions to construct new complexes from existing ones and an extensive library of triangulations of manifolds. For an introduction to the field of piecewise linear (PL) topology see the books [16] and [11].

2 What is new

simpcomp is a package for working with simplicial complexes in the GAP system. In contrast to the package homology [3] which focuses on simplicial homology computation, simpcomp claims to provide the user with a broader spectrum of functionality regarding simplicial constructions.

simpcomp allows the user to interactively construct complexes and to compute their properties in the GAP shell. Furthermore, it makes use of GAP's expertise in groups and group operations. For example, automorphism groups and fundamental groups of complexes can be computed and examined further within the GAP system. Apart from supplying a facet list, the user can as well construct simplicial complexes from a set of generators and a prescribed automorphism group – the latter form being the common in which a complex is presented in a publication. This feature is to our knowledge unique to simpcomp. Furthermore, simpcomp as of Version 1.3.0 supports the construction of simplicial complexes of prescribed dimension, vertex number and transitive automorphism group as described in [14], [2].

Furthermore, simpcomp has an extensive library of known triangulations of manifolds. This is the first time that they are easily accessible without having to look them up in the literature [12], [2], or online [13]. This allows the user to work with many different known triangulations without having to construct them first. As of the current version 1.3.0 the library contains triangulations of roughly 650 manifolds and roughly 7000 pseudomanifolds, including all vertex transitive triangulations from [13]. Most properties that simpcomp can handle are precomputed for complexes in the library. Searching in the library is possible by the complexes' names as well as some of their properties (such as f-, g- and h-vectors and their homology).

3 simpcomp benefits

simpcomp is written entirely in the GAP scripting language, thus giving the user the possibility to see behind the scenes and to customize or alter simpcomp functions if needed.

The main benefit when working with simpcomp over implementing the needed functions from scratch is that simpcomp encapsulates all methods and properties of a simplicial complex in a new GAP object type (as an abstract data type). This way, among other things, simpcomp can transparently cache properties

already calculated, thus preventing unnecessary double calculations. It also takes care of the error-prone vertex labeling of a complex.

simpcomp provides the user with functions to save and load the simplicial complexes to and from files and to import and export a complex in various formats (e.g. from and to polymake/TOPAZ [7], Macaulay2 [8], LaTeX, etc.).

In contrast to the software package polymake [7] providing the most efficient algorithms for each task in form of a heterogeneous package (where algorithms are implemented in various languages), the primary goal when developing simpcomp was not efficiency (this is already limited by the GAP scripting language), but rather ease of use and ease of extensibility by the user in the GAP language with all its mathematical and algebraic capabilities.

The package includes an extensive manual (see [4]) in which all functionality of simpcomp is documented and also makes use of GAP's built in help system such that all the documentation is available directly from the GAP prompt in an interactive way.

4 Some operations and constructions that simpcomp supports

simpcomp implements many standard and often needed functions for working with simplicial complexes. These functions can be roughly divided into three groups: (i) functions generating simplicial complexes (ii) functions to construct new complexes from old and (iii) functions calculating properties of complexes – for a full list of supported features see the documentation [4].

simpcomp furthermore implements a variety of functions connected to *bistellar moves* (also known as *Pachner moves* [15]) on simplicial complexes. For example, simpcomp can be used to construct randomized spheres or randomize a given complex. Another prominent application of bistellar moves implemented in simpcomp is a heuristical algorithm that determines whether a simplicial complex is a *combinatorial manifold* (i.e. that each link is PL homeomorphic to the boundary of the simplex). This algorithm was first presented by F.H. Lutz and A. Björner [1]. It uses a simulated annealing type strategy in order to minimize vertex numbers of triangulations while leaving the PL homeomorphism type invariant.

The package also supports *slicings* of 3-manifolds (known as *normal surfaces*, see [10], [9], [17]) and related constructions.

The current version of simpcomp is 1.3.0 (May 26th, 2010). On the roadmap for the next version 1.4.x which should appear still in 2010 are the support for *simplicial blowups*, i.e. the resolutions of ordinary double points in combinatorial 4-pseudomanifolds. This functionality is to the authors' knowledge not provided by any other software package so far. Also, a closer interaction with the software system Macaulay2 is planned.

5 An example

This section contains a small demonstration of the capabilities of simpcomp in form of an example construction.

M. Casella and W. Kühnel constructed a triangulated K3 surface with minimum number of 16 vertices in [2]. They presented it in terms of the complex obtained by the automorphism group $G \cong AGL(1, \mathbb{F}_{16})$ given by the five generators

$$G = \left\langle \begin{matrix} (1\,2)(3\,4)(5\,6)(7\,8)(9\,10)(11\,12)(13\,14)(15\,16), \\ (1\,3)(2\,4)(5\,7)(6\,8)(9\,11)(10\,12)(13\,15)(14\,16), \\ (1\,5)(2\,6)(3\,7)(4\,8)(9\,13)(10\,14)(11\,15)(12\,16), \\ (1\,9)(2\,10)(3\,11)(4\,12)(5\,13)(6\,14)(7\,15)(8\,16), \\ (2\,13\,15\,11\,14\,3\,5\,8\,16\,7\,4\,9\,10\,6\,12) \end{matrix} \right\rangle,$$

acting on the two generating simplices $\Delta_1 = \langle 2, 3, 4, 5, 9 \rangle$ and $\Delta_2 = \langle 2, 5, 7, 10, 11 \rangle$. It turned out to be a

non-trivial problem to show (i) that the complex obtained is a combinatorial 4-manifold and (ii) to show that it is homeomorphic to a K3 surface as topological 4-manifold.

This turns out to be a rather easy task using simpcomp, as will be shown below. We will fire up GAP, load simpcomp and then construct the complex from its representation given above:

```
gap> LoadPackage("simpcomp");; #load the package
Loading simpcomp 1.1.21
by F.Effenberger and J.Spreer
http://www.igt.uni-stuttgart.de/LstDiffgeo/simpcomp
gap> SCInfoLevel(0);; #suppress simpcomp info messages
gap> G:=Group((1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16),
> (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13,15)(14,16),
> (1,5)(2,6)(3,7)(4,8)(9,13)(10,14)(11,15)(12,16),
> (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16),
> (2,13,15,11,14,3,5,8,16,7,4,9,10,6,12));;
gap> K3:=SCFromGenerators(G,[[2,3,4,5,9],[2,5,7,10,11]]);
[SimplicialComplex

 Properties known: AutomorphismGroup, AutomorphismGroupSize,
                   AutomorphismGroupStructure, AutomorphismGroupTransitivity,
                   Dim, Facets, Generators, Name, VertexLabels.

 Name="complex from generators under group ((C2 x C2 x C2 x C2) : C5) : C3"
 Dim=4
 AutomorphismGroupSize=240
 AutomorphismGroupStructure="((C2 x C2 x C2 x C2) : C5) : C3"
 AutomorphismGroupTransitivity=2

/SimplicialComplex]
```

We first compute the f-vector, the Euler characteristic and the homology groups of K3:

```
gap> K3.F;
[ 16, 120, 560, 720, 288 ]
gap> K3.Chi;
24
gap> K3.Homology;
[ [ 0, [ ] ], [ 0, [ ] ], [ 22, [ ] ], [ 0, [ ] ], [ 1, [ ] ] ]
```

Now we verify that the complex K3 is a combinatorial manifold using the heuristic algorithm based on bistellar moves described above:

```
gap> K3.IsManifold;
true
```

In a next step we compute the parity and the signature of the intersection form of the complex K3:

```
gap> K3.IntersectionFormParity;
0
gap> K3.IntersectionFormSignature;
[ 22, 3, 19 ]
```

This means that the intersection form of the complex K3 is even. It has dimension 22 and signature $19 - 3 = 16$. Furthermore, K3 is simply connected as can either be verified by showing that the fundamental group is trivial or by checking that the complex is 3-neighborly:

```
gap> K3.FundamentalGroup;
<fp group with 105 generators>
gap> Size(last);
1
gap> K3.Neighborliness;
3
```

It now follows from a theorem of M. Freedman [5] that the complex is in fact homeomorphic to a K3 surface because it has the same (even) intersection form. Furthermore, K3 is a tight triangulation [12, 11] as can be verified as follows:

```
gap> K3.IsTight;
#I  SCIsTight: complex is (k+1)-neighborly 2k-manifold and thus tight.
true
```

We can also have a look at the multiplicity vectors of the perfect polyhedral Morse function [11, Sec. 3B] that orders the vertices $v_1, \ldots, v_{16}$ linearly:

```
gap> SCMorseIsPerfect(K3,[1..16]);
true
gap> SCMorseMultiplicityVector(K3,[1..16]);
[ [ 1, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ], [ 0, 0, 1, 0, 0 ],
  [ 0, 0, 2, 0, 0 ], [ 0, 0, 1, 0, 0 ], [ 0, 0, 4, 0, 0 ], [ 0, 0, 3, 0, 0 ],
  [ 0, 0, 3, 0, 0 ], [ 0, 0, 4, 0, 0 ], [ 0, 0, 1, 0, 0 ], [ 0, 0, 2, 0, 0 ],
  [ 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 1 ] ]
```

Finally, instead of constructing the triangulation K3 from scratch, we could have also loaded it directly from the library, saving us a lot of work. We now load the complex from the library and verify that the complex from the library is combinatorially isomorphic to the complex K3 that we constructed – note below the two different searching methods provided by the library:

```
gap> SCLib.SearchByName("K3");
[ [ 551, "K3 surface" ] ]
gap> SCLib.SearchByAttribute("Dim=4 and F[3]=Binomial(F[1],3)");
[ [ 11, "CP^2 (VT)" ], [ 551, "K3 surface" ] ]
gap> M:=SCLib.Load(551);;
gap> M.IsIsomorphic(K3);
true
```

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