`LocalizeRingForHomalg`
Localize Commutative Rings at Maximal Ideals

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Abstract. The objective of this software presentation is to demonstrate a means of homological computation of finitely presented modules over a commutative ring R localized at a maximal ideal $\mathfrak{m}$. This can be achieved by a reduction of the problem of solving linear systems over $R_\mathfrak{m}$ to the same problem over R. An implementation in the GAP-package `LocalizeRingForHomalg` exists as a part of the `homalg`-project.

1. Linear Systems of Equations over Localized Rings

The paper [BLH] describes an axiomatic setup for algorithmic homological algebra of Abelian categories. This is done by exhibiting all existential quantifiers entering the definition of an Abelian category, which for the sake of computability are turned into constructive ones. The abstract approach can be applied *explicitly* to the Abelian category of finitely presented modules over a so-called computable ring R, i.e., a ring which allows algorithmic solving of one-sided (in)homogenous linear systems in the category of finitely presented R-modules.

One way to solve linear systems over a ring is to use an algorithm computing a "distinguished basis" of a module. Fortunately such algorithms exist for many rings of interest, e.g. the Gaussian algorithm, Hermite normal form algorithm, and Gröbner basis algorithms for a wide class of commutative *and* noncommutative rings. Although computing a distinguished basis is the traditional way to solve linear systems, it is only *one* possibility. Indeed, for the localization $R_\mathfrak{m}$ of R at a finitely generated maximal ideal $\mathfrak{m}$ other means do exist:
Homological computations in the Abelian category of finitely presented modules over the local ring $R_\mathfrak{m} := (R\backslash\mathfrak{m})^{-1}R$ can be reduced to computations over R. In particular, one can avoid computing distinguished bases over $R_\mathfrak{m}$. Any $(r \times c)$-matrix over $R_\mathfrak{m}$ can be viewed as numerator-denominator pair (N, d) with $N \in R^{r\times c}$ and $d \in R\backslash\mathfrak{m}$. Furthermore, solving linear systems over $R_\mathfrak{m}$ can simply be done by solving associated systems over R:

Lemma 1.1 (Homogeneous System/Syzygies). *Let* $\mathtt{A} \in R_\mathfrak{m}^{m\times n}$. *Rewrite* $\mathtt{A} = \frac{\tilde{\mathtt{A}}}{\tilde{a}}$ *with* $\tilde{\mathtt{A}} \in R^{m\times n}$ *and* $\tilde{a} \in R\backslash\mathfrak{m}$. *If* $\tilde{\mathtt{X}} \in R^{k\times m}$ *is a matrix of generating syzygies for* $\tilde{\mathtt{A}}$ *over* R*, then the matrix* $\mathtt{X} := \frac{\tilde{\mathtt{X}}}{1}$ *is a matrix of generating syzygies of* $\mathtt{A}$.

This lemma states that computing a generating set for the solutions of a homogeneous linear system over $R_\mathfrak{m}$ is tantamount to the same computation over R. The proof follows directly from the exactness of localization. The following proposition describes how one can find a particular solution for an inhomogeneous linear system over $R_{\mathfrak{m}}$ by solving one over R. Denominators of matrices are w.l.o.g. assumed 1 since multiplication with them is an isomorphism. Let $\mathtt{m} := \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix} \in R^{k\times 1}$ for the maximal ideal $\mathfrak{m} = \langle m_1 \dots m_k \rangle_R$.

Proposition 1.2 (Particular Solution/Submodule membership). *Let* $\mathtt{A} = \frac{\tilde{\mathtt{A}}}{1} \in R_{\mathfrak{m}}^{m\times n}$ *and* $\mathtt{b} = \frac{\tilde{\mathtt{b}}}{1} \in R_{\mathfrak{m}}^{1\times n}$ *with numerator matrices* $\tilde{\mathtt{A}}$ *and* $\tilde{\mathtt{b}}$ over R. *There exists a row matrix* $\mathtt{t} \in R_{\mathfrak{m}}^{1\times m}$ *with* $\mathtt{tA} + \mathtt{b} = 0$ *iff there exists a matrix* $\tilde{\mathtt{s}} \in R^{1\times(m+k)}$ *satisfying*

$$\tilde{\mathtt{s}} \begin{pmatrix} \tilde{\mathtt{A}} \\ \mathtt{m}\tilde{\mathtt{b}} \end{pmatrix} + \tilde{\mathtt{b}} = 0.$$

The proof is *constructive*. Contrary to GRÖBNER bases, the proof of the proposition does *not* provide a normal form, still it yields a particular solution.

For the case of localized polynomials rings the approach presented here can be compared to MORA's algorithm. The latter provides a distinguished basis for localized polynomial rings – in contrast to the approach suggested by Lemma 1.1 and Proposition 1.2. The main advantage of MORA's algorithm making it indispensable is being able to compute HILBERT series and related invariants. On the other hand MORA's algorithm does not scale when applied to large matrices. Our approach is thus suited to get through the intermediate steps of purely homological computations, while MORA's algorithm can then be applied to the smaller result, e.g. to obtain invariants.

2. Implementation and Data Structures

The suggested specification for implementing homological algebra of ABELian categories is realized in the `homalg` project [hpa10], which at its highest level implements homological algebra independent of the underlying ABELian category. It provides routines to compute (co)homology, derived functors, long exact sequences, CARTAN-EILENBERG resolutions, hyper-derived functors, spectral sequences (of bicomplexes) and the filtration they induce on (co)homology, etc.

The programming language of GAP4 [GAP06] is ideally suited to realize this specification. It provides functional and object oriented paradigms, classical method selection, multi-dispatching, and so-called immediate and true-methods, which are used to teach GAP4 how to avoid unnecessary computations by applying mathematical reasoning. Having a clear specification of all computations with matrices allows them to reside outside GAP4, preferably in a system having performant implementations for solving linear systems.

The algorithms for localized rings are implemented in the GAP4-package `LocalizeRingForHomalg` [BLH10]. The implementation is abstract in the sense that *any* commutative computable ring R supported by the `homalg` project can be localized at any of its finitely generated maximal ideals $\mathfrak{m}$, thus providing a new ring $R_{\mathfrak{m}}$ for the `homalg` project. The package additionally includes an interface to the free implementation of MORA's algorithm in SINGULAR [GPS09], which can also be used to solve linear systems over $R_{\mathfrak{m}}$.

Local matrices A are written as a fraction $\frac{\tilde{\mathtt{A}}}{s}$ with numerator matrix $\tilde{\mathtt{A}}$ over R and a single denominator $s \in R\backslash\mathfrak{m}$ and internally stored as this numerator-denominator pair. This data structure enables heuristics, e.g. computation of gcds to cancel fractions and the heuristic use of global computations to prevent introducing new denominators.

3. Example

Example 3.1. Let $R := \mathbb{Q}[x, y, z]$, $\mathfrak{m} = \langle x, y, z\rangle$ maximal in R and W the $R_\mathfrak{m}$-module given by 5 generators satisfying the 6 relations given as rows of the 6×5-matrix below:

```
gap> LoadPackage( "RingsForHomalg" );; LoadPackage( "LocalizeRingForHomalg" );;
gap> R0 := LocalizeAtZero( HomalgFieldOfRationalsInSingular( ) * "x,y,z" );;
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, R0 );;
gap> W := LeftPresentation( wmat );;
```

Compute the purity (=equidimensional) filtration of the $R_\mathfrak{m}$-module W:

```
gap> filt := PurityFiltration( W );
<The ascending purity filtration with degrees [ -3 .. 0 ] and graded parts:
   0:   <A codegree-[ 1, 1 ]-pure rank 2 left module presented
          by 3 relations for 4 generators>
  -1:   <A codegree-1-pure codim 1 left module presented
          by 3 relations for 2 generators>
  -2:   <A zero left module>
  -3:   <A cyclic reflexively pure codim 3 left module presented
          by 3 relations for a cyclic generator>
of
<A non-pure rank 2 left module presented by 6 relations for 4 generators>>
```

One now has a triangular presentation of W compatible with the purity filtration.

References

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