Symbolic Methods for Solving Systems of Linear Ordinary Differential Equations (III)

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Part 3: Applications to Solving Systems with Rational Function Coefficients

Outline

1. Polynomial Solutions
2. Rational Solutions
3. Exponential Solutions
4. Factorization Using Eigenrings
5. Implementations in Maple

Polynomial Solutions

Polynomial Solutions [B99]

Let $K = \mathbb{C}(x)$ and $\vartheta = x\frac{d}{dx}$.

• Linear Differential System :

$$[A] \qquad\qquad \vartheta y = A(x)y,$$

$A(x)$ is an $n \times n$ matrix with entries in K.

• Polynomial Solutions : functions $y \in \mathbb{C}[x]^n$ such that $\vartheta y = Ay$.

• Problem : Given a system $[A]$ to construct the space of polynomial solutions of $[A]$.

- A first important step consists in computing a bound N on the degree of polynomial solutions.

Bound of The Degree of Polynomial Solutions

- A first important step consists in computing a bound N of the degree of polynomial solutions.
- Such a bound can be obtained from the so-called indicial equation (at $x = \infty$) of the system [A].
- But the indicial equation is not immediately apparent for a given system.
- Need to transform the given system to a suitable form called 'simple form' from which the indicial equation can be immediately obtained.
- Every system can be reduced to an equivalent simple one by using the super-reduction algorithm (Part 2 of this tutorial))

Simple Systems

Simple Systems [B99, BP99]

Consider the system

$$[A] \quad \vartheta y = Ay, \;\; A = (a_{i,j}) \in M_n(\mathbb{C}(x)).$$

We are interested in *Frobenius series solutions* in x^{-1} of the form:

$$\hat{y} = \sum_{i=0}^{+\infty} x^{-i-\lambda_0} \hat{y}_i \;\; \lambda_0 \in \mathbb{C}, \; \hat{y}_i \in \mathbb{C}^n, \hat{y}_0 \neq 0.$$

• A polynomial solution of degree N can be viewed as a Frobenius series solution (at $x = \infty$) with exponent $\lambda_0 = -N$.

• Look for a condition on λ_0 in order that $\hat{y}$ be a solution of system $[A]$.

Let

$$\alpha_i = \max_{1\leq j\leq n} (\deg(a_{i,j}), 0), \quad \text{for} \quad 1 \leq i \leq n$$

$$D = diag(x^{-\alpha_1}, \ldots, x^{-\alpha_n}),$$

Multiplying Equation $[A]$ on the left by D we get

$$\mathcal{L}(y) := D(x)\vartheta y - C(x)y = 0, \quad C = DA.$$

where $D(x), C(x) \in \mathbb{C}[[x^{-1}]]$.
Put

$$C = C_0 + \mathrm{O}(x^{-1}), \quad D = D_0 + \mathrm{O}(x^{-1}).$$

Then

$$\mathcal{L}(\hat{y}) = -x^{-\lambda_0}\left((\lambda_0 D_0 + C_0)\hat{y}_0 + \mathrm{O}(x^{-1})\right).$$

If $\mathcal{L}(\hat{y}) = 0$ then $(\lambda_0 D_0 + C_0)\hat{y}_0 = 0$
wich implies

$$\det(C_0 + \lambda D_0) = 0.$$

Indicial Equation of a Simple System

$\Diamond$ To system $[A]$ we associate the polynomial

$$E_\infty(\lambda) := \det\left(C_0 + \lambda D_0\right).$$

- If y is a nonzero polynomial solution of $[A]$ of degree N then $E_\infty(-N) = 0$.
- The degree of polynomial solution can be bounded by the biggest nonnegative integer root of $E_\infty(-\lambda)$.

$\Diamond$ It may happen that $E_\infty(\lambda)$ vanishes identically in which case it is quite useless for our initial purpose. This motivates the following definition

Definition

The system $[A]$ is called simple at $x = \infty$ if $\det\left(C_0 + \lambda D_0\right) \neq 0$ (as a polynomial in λ).
In this case $E_\infty(\lambda)$ is called the indicial polynomial of $[A]$ at $x = \infty$.

An Example of a Non Simple System

$$x\frac{dy}{dx} = Ay, \quad A = \begin{bmatrix} 1 & x^3 \\ 2x^{-1} & 1 \end{bmatrix}.$$

One has

$$D = \begin{bmatrix} x^{-3} & 0 \\ 0 & 1 \end{bmatrix} \text{ and } C = DA = \begin{bmatrix} x^{-3} & 1 \\ 2x^{-1} & 1 \end{bmatrix}.$$

Thus

$$D_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C_0 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

One has

$$\det(C_0 + \lambda D_0) = 0.$$

Hence $[A]$ is not simple at ∞.

Now consider the matrix

$$T = \begin{bmatrix} 0 & x^2 \\ 1 & 0 \end{bmatrix}$$

and put

$$w = Ty$$

then w satisfies the equivalent differential system

$$x\frac{dw}{dx} = \widetilde{A}w$$

where

$$\widetilde{A} = \left(TA + x\frac{dT}{dx}\right) T^{-1} = \begin{bmatrix} 3 & 2\,x \\ x & 1 \end{bmatrix}$$

One can readily verify that $[\widetilde{A}]$ is simple at ∞ and that its indicial polynomial at ∞ is the constant polynomial -2.

$$x\frac{dy}{dx} = \widetilde{A}y, \quad \widetilde{A} = \begin{bmatrix} 3 & 2\,x \\ x & 1 \end{bmatrix}.$$

One has

$$\widetilde{D} = \begin{bmatrix} x^{-1} & 0 \\ 0 & x^{-1} \end{bmatrix} \text{ and } \widetilde{C} = \widetilde{D}\widetilde{A} = \begin{bmatrix} 3x^{-1} & 2 \\ 1 & x^{-1} \end{bmatrix}.$$

Thus

$$\widetilde{D}_0 = 0 \quad \text{and} \quad \widetilde{C}_0 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

One has

$$\det\left(\widetilde{C}_0 + \lambda \widetilde{D}_0\right) = \det(\widetilde{C}_0) = -2.$$

The indicial polynomial at ∞ is constant hence system $[A]$ has no nonzero polynomial solution.

Back to the general case

Theorem (Bark1997)

Given a differential system $[A]$ $\vartheta y = Ay$*, one can construct a nonsingular matrix* T *polynomial in* x *such that the gauge transformation* $w = Ty$ *takes* $[A]$ *into an equivalent system* $[\widetilde{A}]$ $\vartheta w = \widetilde{A} w$ *which is simple at* ∞*.*

• Such a transformation T can be constructed using *super-reduction* algorithm (Hilali and Wazner (1987), Barkatou-Pflügel 2007) (see Part 2 of this tutorial).

Remark. The fact that the transformation T can be chosen polynomial is important: if y is a polynomial solution of $[A]$ $w = Ty$ is a polynomial solution of the equivalent system $[\widetilde{A}]$.

Remarks

- This notion of simple systems extends to the case of finite singularities $\rightarrow$ useful for computing denominators of rational solutions.
- Another application: computation of regular formal solution (Barkatou-Pflügel 1997).

Rational Solutions

Rational Solutions [B99]

Let $K = \mathbb{C}(x)$ and $\vartheta = x\frac{d}{dx}$.

- Linear Differential System :

$$[A] \qquad\qquad \vartheta y = A(x)y,$$

$A(x)$ is an $n \times n$ matrix with entries in K.

- Rational Solutions : functions $y \in K^n$ such that $\vartheta y = Ay$.

- Problem : Given a system $[A]$, to construct the space $\mathcal{S}_A$ of rational solutions of $[A]$.

Let $\mathcal{S}_A$ be the space of rational solutions of $[A]$.

We proceed in two steps :

STEP 1. Construct a universal denominator for $[A]$, i.e. a polynomial (or rational function) $u(x)$ such that

for all $y \in \mathbb{C}(x)$, if $y \in \mathcal{S}_A$ then uy is a polynomial.

STEP 2. If u is a universal denominator for $[A]$ then set

$$w = uy$$

and search for polynomial solutions of the resulting system in w :

$$\vartheta w = (A(x) + u^{-1}\vartheta u \; I_n)w.$$

Computing Denominators of Rational Solutions

Universal Denominator

The problem: Given a differential system

$$[A] \qquad\qquad \vartheta y = A(x)y,$$

to find a rational function u such that
for all $y \in \mathbb{C}(x)$, if $y \in \mathcal{S}_A$ then uy is a polynomial.

Some Facts:

- If $y \in \mathcal{S}_A$ then the finite poles of y are poles of A.
- Given a pole x_0 of A, one can reduce the system $[A]$ to an equivalent system which is simple at $x = x_0$.
- The reduction can be achieved by a polynomial gauge transformation.

- To each point x_0 corresponds an indicial polynomial $E_{x_0}(\lambda) \in \mathbb{C}[\lambda]$
- If y is a nonzero rational solution with a pole of order m at x_0 then $E_{x_0}(-m) = 0$.
- If for some pole x_0 of A the corresponding indicial polynomial has no integer root then $\mathcal{S}_A = \{0\}$.
- For each pole x_0 of A put:

$$m_{x_0} = \min\{\mu \in \mathbb{Z} : \ E_{x_0}(\mu) = 0\}$$

Then

$$u(x) = \prod (x - x_0)^{-m_{x_0}}$$

is a universal denominator for $[A]$.

Exponential Solutions

Exponential solutions

$$y = \exp\left(\int u\right) z$$

with $u \in \mathbb{C}(x)$, $z \in \mathbb{C}[x]^n$.

For $x_0 \in \mathbb{C} \cup \{\infty\}$ define the *singular part* $S_{x_0}(u)$ of u as the principal part of the Laurent series expansion of u at $x = x_0$.

Idea: there exist local exponential part w such that $w = S_{x_0}(u)$

1. Compute all exponential parts of ramification 1 at all singularities (Use algorithms from Part 2)
2. Reconstruct u from
$$u = \sum_{x_0} S_{x_0}(u).$$
Find candidates $\tilde{u}$, do a change of exponential and search for polynomial solutions.

Drawbacks

- Exponential number of combinations to be checked,
- Large algebraic extensions possible (splitting field).
- Can be improved using the approach of Cluzeau and van Hoeij, 2004: reduce mod p to find the “good” combinations!

Example

$$
\begin{matrix}
\frac{-12+3x+3x^2}{(x-1)x^2} & \frac{12}{(x-1)x^2} & \frac{3+6x}{x(x-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2x-4}{(x-1)x^2} & \frac{5x+4}{x^2} & \frac{2x^2+1}{x(x-1)} & \frac{8}{(x-1)x^2} & \frac{2+4x}{x(x-1)} & 0 & 0 & 0 & 0 \\
\frac{3-x}{(x-1)x^2} & \frac{-4}{(x-1)x^2} & \frac{-9+x+x^2}{(x-1)x^2} & 0 & \frac{8}{(x-1)x^2} & \frac{2+4x}{x(x-1)} & 0 & 0 & 0 \\
0 & \frac{4x-8}{(x-1)x^2} & 0 & \frac{4-5x+7x^2}{(x-1)x^2} & \frac{4x^2+2}{x(x-1)} & 0 & \frac{4}{(x-1)x^2} & \frac{1+2x}{x(x-1)} & 0 \\
0 & \frac{3-x}{(x-1)x^2} & \frac{2x-4}{(x-1)x^2} & -\frac{4}{(x-1)x^2} & \frac{-3x-1+3x^2}{(x-1)x^2} & \frac{2x^2+1}{x(x-1)} & 0 & \frac{4}{(x-1)x^2} & \frac{1+\cdots}{x(x-\cdots} \\
0 & 0 & \frac{6-2x}{(x-1)x^2} & 0 & \frac{-8}{(x-1)x^2} & \frac{-6-x-x^2}{(x-1)x^2} & 0 & 0 & \frac{4\cdots}{(x-\cdots} \\
0 & 0 & 0 & \frac{6x-12}{(x-1)x^2} & 0 & 0 & \frac{12-9x+9x^2}{(x-1)x^2} & \frac{6x^2+3}{x(x-1)} & 0 \\
0 & 0 & 0 & \frac{3-x}{(x-1)x^2} & \frac{4x-8}{(x-1)x^2} & 0 & -\frac{4}{(x-1)x^2} & \frac{7-7x+5x^2}{(x-1)x^2} & \frac{4x^2\cdots}{x(x-\cdots} \\
0 & 0 & 0 & 0 & \frac{6-2x}{(x-1)x^2} & \frac{2x-4}{(x-1)x^2} & 0 & \frac{-8}{(x-1)x^2} & \frac{2-5\cdots}{(x-\cdots} \\
0 & 0 & 0 & 0 & 0 & \frac{9-3x}{(x-1)x^2} & 0 & 0 & -\frac{\cdots}{(x-\cdots}
\end{matrix}
$$

Our program finds the solution

$$e^{\int 3x^{-2}+2x^{-1}-3(x-1)^{-1}} \begin{pmatrix} -x^4 \\ -x^4 \\ x^3 \\ -x^4 \\ x^3 \\ -x^2 \\ -x^4 \\ x^3 \\ -x^2 \\ x \end{pmatrix}$$

Factorization Using Eigenring

Definitions

A system $[A]$ $Y' = AY, \;\; A \in M_n(\mathbb{C}(x))$ is called:

- reducible, if it is equivalent (over $\mathbb{C}(x)$) to a system of the form

$$Z' = \begin{pmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,2} \end{pmatrix} Z. \tag{1}$$

- decomposable if $[A]$ is equivalent to a system of the form (1) with $A_{2,1} = 0$.

- irreducible (indecomposable) if it is not reducible (decomposable).

- completely reducible, if it is equivalent to a block–diagonal system

$$T[A] = \mathrm{diag}(A_{1,1}, \dots, A_{s,s})$$

where each system $[A_{i,i}]$, $1 \leq i \leq s$, is irreducible.

The Eigenring Method

This method was introduced by M. Singer (1996) for factoring differential operators over $K = \mathbb{C}(x)$.

Definition: The eigenring $\mathcal{E}(A)$ of a system $[A]$ is the finite dimensional $\mathbb{C}-$*algebra* of all the matrices $T \in M_n(\mathbb{C}(x))$ satisfying the matrix equation

$$T' = AT - TA.$$

• A simple way to compute $\mathcal{E}(A)$ is to convert the above equation into a n^2–dimensional first order linear differential system and search for rational solutions of this system.

Some Properties

- Elements of $\mathcal{E}(A)$ map a solution of $[A]$ to a solution of $[A]$.
- If $T \in \mathcal{E}(A)$ then all its eigenvalues are constant.
- If two systems $[A]$ and $[B]$ are equivalent, their eigenrings $\mathcal{E}(A)$ and $\mathcal{E}(B)$ are isomorphic as $\mathbb{C}-$algebras. In particular, one has $dim_{\mathbb{C}}\mathcal{E}(A) = dim_{\mathbb{C}}\mathcal{E}(B)$

  More precisely If $B = P^{-1}AP - P^{-1}P'$ with $P \in \mathrm{GL}(n, \mathbb{C}(x))$ then

  $$\mathcal{E}(A) = P^{-1}\mathcal{E}(B)P := \{P^{-1}TP \mid T \in \mathcal{E}(B)\}.$$

- If $[A]$ is decomposable then $\dim_{\mathbb{C}} \mathcal{E}(A) > 1$.

Factorization of Systems with Nontrivial Eigenring

Theorem 1 If $\dim_{\mathbb{C}} \mathcal{E}(A) > 1$ then $[A]$ is reducible and the reduction can be carried out by a matrix $P \in \mathrm{GL}(n, K)$ that can be computed explicitly.

Cor. Given a system $[A]$ one can construct an equivalent matrix equation $[B]$ having a block-triangular form

$$\begin{pmatrix} B_{1,1} & 0 & & 0 \\ B_{2,1} & B_{2,2} & & \\ \vdots & & \ddots & 0 \\ B_{s,1} & \dots & & B_{s,s} \end{pmatrix}$$

where s is the maximal possible, i.e. for each $1 \leq i \leq s$, the eigenring of $[B_{i,i]}$ is trivial (having dimension 1).

Proof of Theorem1

Suppose $\dim \mathcal{E}(A) > 1$. Then there is $T \in \mathcal{E}(A)$ with rank $r < n$. One can compute $P \in GL(n, K)$ such that

$$S := P^{-1}TP = \begin{pmatrix} S_{1,1} & 0 \\ S_{2,1} & 0 \end{pmatrix},$$

where $S_{1,1}$ is an $r \times r$ matrix and $\begin{pmatrix} S_{1,1} \\ S_{2,1} \end{pmatrix}$ has rank r.

Let $B = P^{-1}(AP + P')$ then $S \in \mathcal{E}(B)$, Decompose B in the same form as S

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}.$$

The equation $SB - BS = S'$ implies

$$\begin{pmatrix} S_{1,1} \\ S_{2,1} \end{pmatrix} B_{1,2} = 0.$$

Since $\begin{pmatrix} S_{1,1} \\ S_{2,1} \end{pmatrix}$ is of full rank, then $B_{1,2} = 0$.

Factorization of Decomposable Systems

Proposition Suppose that $\mathcal{E}(A)$ contains an element T which has $s \geq 2$ distinct eigenvalues $\lambda_1, \ldots, \lambda_s \in \mathbb{C}$ then $[A]$ is decomposable.

Moreover, if $P \in GL(n, K)$ is such that

$$J = P^{-1}TP = \bigoplus_{i=1}^{s} J_i \qquad \text{with } \; spec(J_i) = \lambda_i$$

Then the matrix $B = P[A] = P^{-1}(AP + P')$ has the form

$$P[A] = \bigoplus_{i=1}^{s} B_i.$$

Example

$$A = \begin{bmatrix} 9 & -6\,x^{-2} & 0 & 6\,x^{-2} & 6\,x^{-2} \\ \frac{1-x}{x^2} & \frac{4\,x^2-9\,x+4}{x^2-x^3} & 6\,\frac{x-1}{x^2} & \frac{-3+3\,x-4\,x^2+4\,x^3}{x^4} & 4\,\frac{x-1}{x^2} \\ 0 & 5\,\left(x^4-x^3\right)^{-1} & \frac{5-x}{x^2} & -3\,x^{-2} & 5\,x^{-3} \\ 0 & (1-x)^{-1} & 0 & 3\,x^{-3} & -1 \\ x^{-2} & \frac{x^2+5\,x-4}{x^3-x^2} & 6\,\frac{x-1}{x^2} & -\frac{3+4\,x^2}{x^4} & \frac{x^2-4}{x^2} \end{bmatrix}$$

A basis of $\mathcal{E}(A)$ is (I_5, T) where

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -x^{-1} & 0 & 1/2\,\frac{-2\,x+2}{x} & 1/2\,\frac{-2\,x+2}{x} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -x^{-1} & 0 & x^{-1} & 1/4\,\frac{-4\,x+4}{x} \end{bmatrix}$$

Let

$$P = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ \frac{x-1}{x} & 0 & \frac{1+x}{x} & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ -x^{-1} & 0 & x^{-1} & 0 & 0 \end{bmatrix}$$

Then

$$J := P^{-1}TP = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

As expected

$$P[A] = B_1 \oplus B_2$$

where

$$B_1 = \begin{bmatrix} -5\,x^{-1} & x^{-1} \\ -\frac{9\,x^2+5\,x-6}{x^2} & \frac{9\,x+1}{x} \end{bmatrix}$$

and

$$B_2 = \begin{bmatrix} \frac{-10\,x^3+8\,x^4+6\,x^2-3\,x+3}{x^3(x-1)} & -\frac{x^4+x^3+3-3\,x}{x^3(x-1)} & 6\,\frac{x-1}{x} \\ 2\,\frac{3-3\,x+3\,x^2-4\,x^3+4\,x^4}{x^3(x-1)} & -\frac{2\,x^3-6\,x+6+x^4}{x^3(x-1)} & 6\,\frac{x-1}{x} \\ -\frac{-7-x+3\,x^2-9\,x^3+8\,x^4}{x^3(x-1)} & \frac{x^3+3\,x^2-6\,x-2+x^4}{x^3(x-1)} & -\frac{-5\,x-5+6\,x^2}{x^2} \end{bmatrix}$$

Implementation

ISOLDE – Integration of Systems of Ordinary Linear Differential Equations

• Implemented in Maple. Available at:
http://sourceforge.net/projects/isolde.

• ISOLDE implements algorithms by E. Pflügel and M. Barkatou for solving systems of first-order linear ODE's:

$$Y' = AY + b \tag{2}$$

where $A \in M_n(K)$ and $b \in K^n$.

$$K = \mathbb{C}((x)) = \mathbb{C}[[x]][x^{-1}], \quad \text{or} \quad K = \mathbb{C}(x), \quad ' = \frac{d}{dx}$$

• The main approach is a direct treatment of the system (we avoid *cyclic vectors*).

Conclusion

Recent and Current Developments - Perspectives

- **Recent Past**:
  - Modular Algorithms for Linear Differential Equations: PhD Thesis of Thomas Cluzeau (Limoges 2004)
  - Formal reduction of pfaffian systems: PhD Thesis of Nicolas LeRoux (Limoges 2006)
- **Current**:
  - Algorithms for solving directly systems of higher order differential equations: Thesis of Carole El Bacha
  - Reduced Forms of Linear Differential Systems and Applications to Integrability of Hamiltonian Systems : Thesis of Ainhoa Aparicio
  - Extension to DAE's (work in progress, collaboration with E. Pflügel).

Other Works - Perspectives

- **Other Works**
  - 'EG Elemination' Approach: S. Abramov and his students
  - 'Levelt' Approach: E. Corel (2002) (Framework: Lattices and Linear Connections over Vector Spaces)
  - Symbolic-Numeric Methods: J. van der Hoeven (2004)
- **Future Works:**
  - Complexity Analysis
  - Modular Approach for Matrix Case
  - Extension to Integrable Systems of Linear PDE's
  - Develop Specific Reduction Algorithms for Hamiltonian Systems
  - Extension to Non Linear Systems
  - Matrix Differential Equations with Parameters
  - ...

References

See the abstract of this tutorial in the Proceedings of ISSAC'10.