Symbolic Methods for Solving Systems of Linear Ordinary Differential Equations (II)

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Part 2 : Systems of Second Kind - Fundamental Algorithms

Outline

1. How to recognize a regular singularity ? Moser Algorithm
2. Splitting Lemma
3. Katz Invariant
4. Formal Reduction Algorithm
5. Formal Solutions

Systems of Second Kind

A matrix linear differential equation with Poincaré rank $p > 0$:

$$[A] \qquad\qquad Y' = AY,$$

$$A(x) = \frac{1}{x^{p+1}} \sum_{i=0}^{\infty} A_i x^i, \quad A_i \in M_n(\mathbb{C}), \;\; A_0 \neq 0.$$

- System $[A]$ has a regular singularity at $x = 0$ if it is equivalent to a system of the first kind (for which $p = 0$) (see Part 1).
- Problem: Give an algorithm to decide for any system of second kind whether it has regular singularity.

Systems of Second Kind with Regular Singularity

How to recognize a regular singular system?

Problem 1: Given a system $[A]$ of second kind, i.e. with Poincaré rank $p(A) > 0$, to decide whether it is regular singular or not.

In other words, how to decide if the Poincaré rank of the given system can be reduced to 0 or not?

Problem 2: Given a system $[A]$ with Poincaré rank $p(A) > 0$, to decide whether there exists $T \in \mathrm{GL}(n, K)$ such that $p(T[A]) < p(A)$.

There is an algorithm due to Moser (1960) which transforms a given system $[A]$ to an equivalent one with minimal Poincaré rank.

Other methods for reducing Poincaré rank (to its minimal value): Levelt (1992), Wagenfurer (1989), ..., Corel (2003).

Moser Reduced Systems

$$A(x) = \frac{1}{x^{p+1}} \sum_{i=0}^{\infty} A_i x^i, \quad A_i \in M_n(\mathbb{C}), \quad A_0 \neq 0, \; p \in \mathbb{Z}.$$

Moser rank: $m(A) = p + \frac{rank(A_0)}{n}$ if $p > 0$, otherwise $m(A) = 1$.

Moser invariant: $\mu(A) = \min\{m(T[A]) \mid T \in GL(n, \mathbb{C}((x)))\}$

Definition. $[A]$ is said to be Moser-reducible if $m(A) > \mu(A)$.

- $[A]$ is Moser-reducible $\iff \exists T \in GL(n, \mathbb{C}((x)))$ such that $m(T[A]) < m(A)$.

- $x = 0$ is regular singular for $[A] \iff \mu(A) = 1$.

A Criterion for Moser-reducibility

Theorem. [Moser 1960]

1. If $p > 0$ then A is Moser-reducible iff the polynomial

$$\Theta_A(\lambda) := x^{rank(A_0)} \det\left(\lambda I - A_0/x - A_1\right)_{|x=0} \equiv 0.$$

2. If A is Moser reducible then the reduction can be carried out with a transformation of the form
$T = (P_0 + xP_1) diag(1, \ldots, 1, x, \ldots, x), \quad P_i \in \mathbb{C}^{n\times n}, det P_0 \neq 0.$

- Applying Moser's Theorem several times, if necessary, $\mu(A)$ can be determined.

- Further, a matrix polynomial $T \in GL(n, K)$ such that $m(T[A]) = \mu(A)$ can be computed in this way

Remarks

- Moser's initial intention: *classification of singularity*
- Barkatou (1997): also useful for computing formal solutions in the irregular singular case.
- Moser's Theorem can be applied to a system $[A]$ for diminishing the number $p(A)$, when it is possible.
- A necessary condition that there exist a gauge transformation $T \in GL(n, \mathbb{C}((x)))$ such that $T[A] = \frac{1}{x^{p'+1}}(B_0 + B_1 x + \cdots)$ with $p' < p$ $(B_0 \neq 0)$, is that A_0 is nilpotent.

Review: Moser Reduction Algorithms

- There are various algorithms to compute T such that $T[A]$ is Moser-reduced.
- Moser's paper: no constructive algorithm given
- Dietrich (1978), Hilali/Wazner (1987): first efficient algorithms,
- Barkatou (1995): version for rational function coefficients, implemented in ISOLDE
- Barkatou-Pflügel (2007): New reduction algorithm + complexity analysis.

Description of Moser Algorithm

- By a constant gauge transformation we can put A_0 in the form:

$$A_0 = \begin{pmatrix} A_0^{11} & 0 \\ A_0^{21} & 0 \end{pmatrix}, \;\; A_0^{11} \in \mathbb{C}^{r\times r} \;\; r = rank(A_0).$$

- Let A_1 be partitioned so that A_1^{11} is a square matrix of order r:

$$A_1 = \begin{pmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{pmatrix},$$

- Consider

$$G_\lambda(A) = \begin{pmatrix} A_0^{11} & A_1^{12} \\ A_0^{21} & A_1^{22} + \lambda I_{n-r} \end{pmatrix}.$$

- Then $\det G_\lambda(A)) = \Theta_A(\lambda)$.
- A is Moser-reducible $\iff$ $\det G_\lambda(A) \equiv 0$.

Case 1: $rank(A_0^{11} \;\; A_1^{12}) < r$

A is Moser-reducible $\iff \begin{vmatrix} A_0^{11} & A_1^{12} \\ A_0^{21} & A_1^{22} + \lambda I_{n-r} \end{vmatrix} = 0.$

Proposition 1 If $m(A) > 1$ and $rank(A_0^{11} \;\; A_1^{12}) < r$, then A is reducible and the reduction can be carried out with the gauge transformation

$$T = diag(xI_r, I_{n-r}).$$

Proof: Let $B = T[A] = T^{-1}AT - T^{-1}\frac{dT}{dx}$.

$$B = x^{-p-1}[B_0 + xB_1 + \cdots] + x^{-1}diag(I_r, 0)$$

where

$$B_0 = \begin{pmatrix} A_0^{11} & A_1^{12} \\ 0 & 0 \end{pmatrix},$$

Since $p > 0$, then $m(B) = p + rank(B_0)/n < m(A) = p + r/n$.

Example

$$A = x^{-2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} + x^{-1}\begin{pmatrix} 4 & 0 \\ 0 & -3 \end{pmatrix} + \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}.$$

Here $p = 1,\ r = 1 \Rightarrow m(A) = 1 + 1/2 = 3/2 > 1$.

$\det G_\lambda(A) = \begin{vmatrix} 0 & 0 \\ 2 & -3+\lambda \end{vmatrix} = 0 \Rightarrow A$ is Moser-reducible.

Let

$$T = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

$$B := T[A] = T^{-1}AT - T^{-1}T' = \frac{1}{x}\begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}.$$

The system $Z' = BZ$ has a singularity of first kind at $x = 0$.

Hence $Y' = AY$ has a regular singularity at $x = 0$.

To solve $Y' = AY$, it suffices to solve $Z' = BZ$ whose solution can be obtained immediately since $B = x^{-1}B_0$ where B_0 is the constant matrix:

$$B_0 = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}.$$

The matrix B_0 is diagonalizable:

$$B_0 = P^{-1}JP, \quad \text{où} \quad P = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \quad \text{et} \quad J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\Rightarrow Px^J$ is a fundamental solution matrix for $Z' = BZ$.

It follows that

$$W = TPx^J = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} -1 & 2\,x^2 \\ -x^{-1} & x \end{pmatrix}$$

is a fundamental solution matrix of $Y' = AY$.

Case 2: $rank(A_0^{11} \;\; A_1^{12}) = r$

Proposition 2 If A is reducible and $rank(A_0^{11} \;\; A_1^{12}) = r$, then there exists a constant matrix Q such that the matrix $G_\lambda(Q[A])$ has the form has the following particular form:

$$G_\lambda(A) = \begin{pmatrix} A_0^{11} & U_1 & U_2 \\ V_1 & W_1 + \lambda I_{n-r-h} & W2 \\ 0 & 0 & W_3 + \lambda I_h \end{pmatrix}, \qquad (1)$$

where $1 \leq h \leq n-r$, $W1, \; W3$ are square matrices of order $(n-r-h)$ and h respectively, W_3 is upper triangular with zero diagonal with the condition

$$rank(A_0^{11} \;\; U_1) < r \qquad (2)$$

Proposition 3 If $m(A) > 1$ and $G_\lambda(A)$ has the form (1) with the condition (2), then A is reducible and the reduction can be carried out with the transformation

$$T = diag(xI_r, I_{n-r-h}, xI_h)$$

Proof: Put $B = T[A] = T^{-1}AT - T^{-1}\frac{dT}{dx}$. One has

$$B = x^{-p-1}[B_0 + xB_1 + \cdots] + x^{-1}\mathrm{diag}(I_r, 0, I_h)$$

where

$$B_0 = \begin{pmatrix} A_0^{11} & U_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and then $rank(B_0) = rank(A_0^{11} \;\; U_1) < r = rank(A_0)$. On the other hand since $p > 0$, then $m(B) = p + rank(B_0)/n$. Hence $m(B) < m(A)$.

Example

Consider the system $[A]\ \ \frac{dY}{dx} = A(x)Y$

$$A(x) = \begin{bmatrix} -2\,x^{-1} & 0 & x^{-2} & 0 \\ x^2 & -\frac{-1+x^2}{x} & x^2 & -x^3 \\ 0 & x^{-2} & x & 0 \\ x^2 & x^{-1} & 0 & -\frac{x^2+1}{x} \end{bmatrix}$$

Here

$$p = 1, \quad r = rank(A_0) = 2.$$

Hence

$$m(A) = 1 + 2/4 = 3/2 > 1.$$

One can check that

$$\Theta_A(\lambda) := x^{rank(A_0)} \det\left(\lambda I - A_0/x - A_1\right)_{|x=0} \equiv 0$$

Hence A is Moser reducible.

The equivalent matrix B computed by our implementation is

$$B(x) = \begin{bmatrix} -\frac{x^2+1}{x} & x & 1 & -x \\ x^{-1} & \frac{-1+x^2}{x} & 0 & 0 \\ 0 & x^{-1} & -2\,x^{-1} & 0 \\ x & 0 & x^2 & -\frac{x^2+1}{x} \end{bmatrix}$$

The transformation T is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ x^2 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence $[A]$ is singular regular.

Consider the system $\frac{dY}{dx} = x^{-1}A(x)Y$ where

$$A(x) = \begin{pmatrix} 4 & x^3 & -2x^6 & -x^6 \\ 0 & -1-x^{-1} & x^{-1} & 0 \\ x^{-7} & 0 & x^{-1}-2 & x^{-1} \\ x^{-5}+x^{-6} & -x^{-2} & x^2+x+x^{-2} & -3 \end{pmatrix}$$

Here $m(A) = 7 + 1/4 = 29/4$.

$$x^{-1}B(x) = \begin{pmatrix} -2-x^{-1} & 0 & x^{-1} & 0 \\ x^{-2}-x^{-1} & x-1 & x^3+x^2-2x+x^{-1} & -3-x \\ 0 & x^{-2} & x^{-1}-3 & 0 \\ -x^{-1} & x+1 & x^3+x^2+x^{-1} & -x-4 \end{pmatrix}$$

The transformation T is

$$\begin{pmatrix} 0 & x^6 & 0 & -x^6 \\ x & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

One has $\mu(A) = m(B) = 2 + 2/4 = 5/2$.

Systems of Second Kind with Irregular Singularity

Formal Solutions

Consider a system $[A]$ $Y' = AY$ with minimal Poincaré rank $p > 0$:

$$A(x) = \frac{1}{x^{p+1}} \sum_{i=0}^{\infty} A_i x^i, \quad A_i \in M_n(\mathbb{C}), \;\; A_0 \neq 0.$$

System $[A]$ has a formal fundamental solution matrix of the form

$$\Phi(x^{1/s}) x^{\Lambda} \exp\left(Q(x^{-1/s})\right)$$

$s \in \mathbb{N}^*$, $\Phi \in \mathrm{GL}(n, \mathbb{C}((x^{1/s})))$,
$Q(x^{-1/s}) = \mathrm{diag}\left(q_1(x^{-1/s}), \ldots, q_n(x^{-1/s})\right)$
the q_i's are polynomials in $t = x^{-1/s}$ over $\mathbb{C}$ without constant term
Λ is a constant matrix commuting with Q.

- The smallest possible s is called the degree of ramification of $[A]$.

How to compute the formal solutions of $[A]$?

Formal Reduction : an algorithmic procedure that allows construction of formal solutions.

Main idea: Transformation of system into new system with smaller p or n

Important tools: Moser Algorithm, Splitting Lemma, Katz Invariant computation.

- Discussion depending on structure of A_0. We distinguish two cases:
  1. Case 1: A_0 has at least two eigenvalues.
  2. Case 2: A_0 has only one eigenvalue.

Case 1- The Splitting Lemma

Splitting Lemma

Theorem: Consider a system $[A]$

$A(x) = x^{-p-1} \sum_{i=0}^{\infty} A_i x^i, \quad A_0 \neq 0, \;\; p > 0$ and assume that A_0 is block-diagonal

$$A_0 = \begin{pmatrix} A_0^{11} & 0 \\ 0 & A_0^{22} \end{pmatrix} \text{ with } \operatorname{spec}(A_0^{11}) \cap \operatorname{spec}(A_0^{22}) = \emptyset.$$

Then there exists a gauge transformation of the form

$$T(x) = \sum_{j=0}^{\infty} T_j x^j \quad (T_0 = I)$$

such that the matrix $B := T[A]$ is block-diagonal matrix with the same block partition as in A_0

$$B = x^{-p-1} \begin{pmatrix} B^{11}(x) & 0 \\ 0 & B^{22}(x) \end{pmatrix}.$$

Sketch of Proof

- Put $T_0 = I$ and $B_0 = A_0$

- Look for matrices T_i of the special form

$$T_i = \begin{pmatrix} 0 & T_i^{12} \\ T_i^{21} & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} B_i^{11} & 0 \\ 0 & B_i^{22} \end{pmatrix}.$$

- Then for $i \geq 1$ the coefficients T_i and B_i can be obtained by successively solving Sylvester linear equations of the form

$$A_0^{11} X - X A_0^{22} = U_i \text{ or } A_0^{22} Y - Y A_0^{11} = V_i$$

where U_i and V_i depend only on A_j, B_j, T_j for $j = 0, \ldots, i-1$.

A very simple situation

$$A(x) = x^{p-1} \sum_{i=0}^{\infty} A_i x^i, \quad A_0 \neq 0, \; p > 0.$$

Corolary. If A_0 has all distinct eigenvalues, then there exists $T \in \mathrm{GL}(n, \mathbb{C}[[x]])$ such that $T[A]$ is a diagonal matrix.

If $B_0 := P^{-1} A_0 P = \mathrm{diag}(\beta_1, \ldots, \beta_n)$ with $\beta_i \neq \beta_j$ for $i \neq j$ for some $P \in \mathrm{GL}(n, \mathbb{C})$, then there exists a formal transformation

$$T(x) = \sum_{j \geq 0} T_j x^j \quad (T_0 = P)$$

such that

$$T[A] = \begin{pmatrix} \frac{\beta_1}{x^{p+1}} + O(\frac{1}{x^p}) & & 0 \\ & \ddots & \\ 0 & & \frac{\beta_n}{x^{p+1}} + O(\frac{1}{x^p}) \end{pmatrix}$$

Case 2- The Nilpotent Case

Reduction to the Case where A_0 is Nilpotent

Let

$$A(x) = \frac{1}{x^{p+1}} \sum_{i=0}^{\infty} A_i x^i, \quad A_0 \neq 0,\ p > 0.$$

- Apply the Splitting Lemma to decouple $[A]$ along the spectral subspaces of A_0:

  $$A = A^{(1)} \oplus \cdots \oplus A^{(k)}$$

  The leading matrix of each subsystem has only one eigenvalue.

- If $A_0 = \alpha I \oplus N$, with N nilpotent then apply the substitution $Y = \exp\left(\frac{-\alpha}{p x^p}\right) Z$ which replace A by $A - \dfrac{\alpha}{x^{p+1}} I$.
  This makes A_0 nilpotent.

- If necessary, apply the Moser algorithm to replace the system by an equivalent one with minimal Poincaré rank p.

The case A_0 nilpotent and $p > 0$ minimal

- In this case we need algebraic extension of K:

  Gauge transformations in $\mathbb{C}((x^{1/m}))$, for suitable integer $m \geq 2$, are applied to get an equivalent system $[\widetilde{A}]$ with leading coefficient $\widetilde{A}_0$ having distinct eigenvalues.

- How to choose m?

  Compute κ, the *Katz invariant* of $[A]$ (see below) and let m be the smallest positive integer such that $m\kappa$ is an integer.

- Using Moser Algorithm yields a system with Poincaré rank equal to $m\kappa$ and leading matrix A_0 with at least two eigenvalues.

- So we can again split the problem into problems of lower size, and so on.

Katz Invariant

Katz Invariant

Definition: The Katz Invariant of $[A]$ is the rational number

$$\kappa(A) = \max_{1 \le j \le n} \deg_x(q_j)$$

where the q_j are the entries of the exponential part Q of $[A]$.

Fact: $\kappa(A) \le p(A)$ with equality iff A_0 is non-nilpotent.

Theorem[Bark05] Suppose A Moser reduced and A_0 nilpotent. Then

$$p(A) - 1 + \frac{r}{n-d} \le \kappa(A) \le p(A) - \frac{1}{n-d}$$

where $r = rank(A_0)$ and $d = deg\,\Theta_A(\lambda)$.

Example 1

$$A(x) = \frac{1}{x^4}\begin{bmatrix} 0 & 0 & x & 0 \\ 1 & -x^2 & x^2 & -x^2 \\ 0 & 1 & x^2 & 0 \\ x^2 & x^2 & 0 & -x^2 \end{bmatrix}$$ has Poincaré rank $p(A) = 3$.

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is nilpotent and has rank $r = 2$.

$\Theta_A(\lambda) = \lambda$ is not zero and has degree $d = 1$.

The above theorem tells us:

$$2 + 2/3 = p(A) - 1 + \frac{r}{n-d} \leq \kappa(A) \leq p(A) - \frac{1}{n-d} = 3 - 1/3.$$

Hence $\kappa(A) = 8/3$.

How to obtain Katz Invariant?

If $[A]$ is Moser-reduced and its leading coefficient A_0 is nilpotent then $\kappa(A)$ is not an integer.

Theorem[Bark05] Let A be Moser-reduced, put $r = rank(A_0)$, $d = deg(\Theta_A(\lambda))$ and write

$$\det(\lambda I - A(x)) = \lambda^n + a_{n-1}(x)\lambda^{n-1} + \cdots + a_0(x).$$

Suppose

$$\text{(C)} \qquad p(A) \geq (1 - \frac{r}{n-d})(r+1).$$

Then

$$\kappa(A) = \max\left(0, \max_{0 \leq j < n}\left(\frac{-n + j - \mathrm{val}(a_j)}{n-j}\right)\right)$$

Back to Example 1

$[A]$ is Moser-reduced, $p = 3,\ r = 2,\ d = 1.$

Condition (C) in the above theorem is satisfied.

One can compute $\kappa(A)$ using the above formula:

$$\det(\lambda I - A(x)) = \lambda^4 + \frac{\lambda^3}{x^2} - \frac{\lambda^2}{x^6} + \frac{\left(-2\,x^7 - x^2 - x^5\right)\lambda}{x^{13}} + \frac{x^2 - 1}{x^{13}}.$$

One has

$$\mathrm{val}(a_3) = -2,\ \mathrm{val}(a_2) = -6,\ \mathrm{val}(a_1) = -11,\ \mathrm{val}(a_0) = -13.$$

Hence

$$\kappa(A) = \max\left(0, \max_{0\le j<n}\left(\frac{-n + j - \mathrm{val}(a_j)}{n - j}\right)\right) = \max\left\{0, 1, 2, \frac{8}{3}, \frac{9}{4}\right\} = \frac{8}{3}.$$

Remarks

• It is always possible to come down to the case where Condition (C) is fulfilled.

Idea: If $p(A) < (1 - \frac{r}{n-d})(r+1)$, use a ramification $x = t^s$ where

$$s \geq \frac{n-r-d}{p-2+r/(n-d)}$$

• The following conjecture is likely to be true.

Conjecture: No need of Condition (C) in the above theorem.

What do we gain by computing Katz invariant?

Suppose $[A]$ be Moser-reduced and A_0 nilpotent and let $\kappa(A) = \frac{\ell}{m}$ with $(\ell, m) \in \mathbb{N} \times \mathbb{N}$ with $gcd(\ell, m) = 1$.

Put $t = x^{1/m}$ and let $[\widetilde{A}]$ denote the resulting system:

$$\frac{dY}{dt} = \widetilde{A}Y, \quad \widetilde{A}(t) = mt^{m-1}A(t^m).$$

Then there is a $T \in GL(n, \mathbb{C}((t)))$ such that

- $\widetilde{B} := T[\widetilde{(A)}]$ has Poincaré rank equal to ℓ
- its leading matrix $\widetilde{B}_0$ has at least m distinct eigenvalues.

Remark The transformation T is in fact polynomial in t and can be computed using Moser Algorithm.

Back to our example

We have

$$\kappa(A) = \frac{8}{3}.$$

The change of variable

$$x = t^3$$

yields

$$\frac{dY}{dt} = \widetilde{A}(t)Y$$

where

$$\widetilde{A}(t) = \frac{3}{t^{10}} \begin{bmatrix} 0 & 0 & t^3 & 0 \\ 1 & -t^6 & t^6 & -t^6 \\ 0 & 1 & t^6 & 0 \\ t^6 & t^6 & 0 & -t^6 \end{bmatrix}.$$

One can check that this system is not Moser-reduced.

Moser Algorithm produces the gauge transformation

$$Y = SZ$$

where

$$S = \mathrm{diag}(t^2, t, 1, 1),$$

and the equivalent system

$$\frac{dZ}{dt} = \widetilde{B}(t)Z, \quad \widetilde{B}(t) = \frac{1}{t^9}\begin{bmatrix} -2\,t^8 & 0 & 3 & 0 \\ 3 & -3\,t^5 - t^8 & 3\,t^4 & -3\,t^4 \\ 0 & 3 & 3\,t^5 & 0 \\ 3\,t^7 & 3\,t^6 & 0 & -3\,t^5 \end{bmatrix}$$

Its Poincaré rank is equal to 8 as expected.

The leading matrix is

$$\widetilde{B}_0 = \begin{bmatrix} 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is non nilpotent and has 4 distinct eigenvalues

$$0, 3, 3j, 3j^2$$

with $j^3 = 1$.

The system can be then decoupled into 4 scalar equations.

One formal fundamental solution can be written as

$$\widehat{Y}(x) = \widehat{F}(x) \begin{bmatrix} e^{1/x} & 0 \\ 0 & x^J U e^{Q(1/x)} \end{bmatrix}$$

where $\widehat{F}(x)$is a meromorphic formal series in x,

$$J = -\frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 1 & j & j^2 \\ 1 & j^2 & j^4 \end{bmatrix}$$

and

$$Q(1/x) = \begin{bmatrix} q(1/x) & 0 & \\ 0 & q(1/(jx)) & 0 \\ 0 & 0 & q(1/(j^2x)) \end{bmatrix}$$

with

$$q\left(\frac{1}{x}\right) = \frac{-3}{8x^{8/3}} - \frac{1}{4x^{4/3}}.$$

Summary: Formal Reduction Algorithm

Algorithm

1. $p := p(A)$; $n := dim(A)$; $A_0 :=$lead-mat(A);
2. if A_0 is nilpotent then make A Moser-reduced (p minimal)
3. if $n = 1$ or $p = 0$ then STOP
4. if $p > 0$ and $n > 1$ and A_0 nilpotent then
   - 4.1 compute $\kappa(A) = \frac{\ell}{m}$, $gcd(\ell, m) = 1$
   - 4.2 replace x by x^m
   - 4.3 Make A Moser-reduced
5. if A_0 is not nilpotent then for each nonzero eigenvalue α of A_0 do
   - 5.1 p:=p(A); $A := A - \frac{\alpha}{x^{p+1}} I$
   - 5.2 Apply the Splitting Lemma to get $B := diag(B^{(1)}, B^{(2)})$ with $B_0^{(2)}$ nilpotent.
   - 5.3 $A := B^{(2)}$ and go to Step 1

▷ At each step, one either reduces the Poincaré rank p or the order n of the system.

$\Rightarrow$ after a finite number of steps one either has $p = 0$ or $n = 1$.

An Important Question

Given a matrix

$$A(x) = x^{-p-1}(A_0 + A_1 x + \cdots), \qquad p > 0$$

• **Question:** How many terms in $\sum_{i=0}^{\infty} A_i x^{i-p-1}$ are necessary for computing the exponential part $Q(x^{-1/s})$ of the system$[A]$?

• The answer can be found in Lutz-Schäfke (1985) or Babbit-Varadarajan (1983):
The exponential part $Q(x^{-1/s})$ is determined by the coefficients

$$A_0, A_1, \cdots, A_{np-1}$$

Example

$$A = \begin{bmatrix} -\frac{5}{x^2} & \frac{5}{x^2} & -x^{-3} & \frac{4}{x^2} \\ 0 & \frac{1-4x}{x^3} & -x^{-2} & -\frac{2}{x^2} \\ \frac{2x+1}{x^3} & \frac{1-5x}{x^3} & \frac{2-3x}{x^3} & \frac{1-4x}{x^3} \\ 0 & \frac{4}{x^2} & x^{-2} & \frac{1+x}{x^3} \end{bmatrix}$$

$> Rational\_Exponential\_Part(A, x);$

$$[x = \alpha t^2, \frac{-\frac{9}{8} + \frac{7\alpha}{4}}{t} + \frac{-1/4 - \frac{11\alpha}{4}}{t^2} + \frac{2\alpha}{3t^3} + \frac{1}{2t^4}]$$

where

$$\alpha = RootOf(\_Z^2 + 1)$$

Super-irreducible Forms

Super–irreducible Forms

- Consider a system $[A]$ $Y' = AY$ with $p > 0$.
- For $k = 1, \dots, p$, put

$$m_k(A) = p + \frac{n_0}{n} + \frac{n_1}{n^2} + \cdots + \frac{n_{k-1}}{n^k}$$

where $n_i = \#$ of rows of A with valuation $-p + i$.

- Define

$$\mu_k(A) = \min\{m_k(T[A]) \mid T \in \mathrm{GL}(n, K)\}.$$

- The matrix A is said to be k-irreducible if $m_k(A) = \mu_k(A)$. Otherwise A is called k-reducible.
- The matrix A is said to be super–irreducible, if it is k-irreducible for every k, or equivalently if

$$m_p(A) = \mu_p(A).$$

A Criterion for $k-$reducibility

- One defines

$$s_k := kn_0 + (k-1)n_1 + \cdots + n_{k-1}$$

and

$$\Theta_k(\lambda) := x^{s_k} \det(x^{p-k}A - \lambda I_n)$$

- One verifies that $\Theta_k(\lambda)$ belongs to $\mathbb{C}[[x]][\lambda]$.

- One can define then the polynomial $\theta_k(\lambda) \in \mathbb{C}[\lambda]$ as

$$\theta_k(\lambda) = \left(x^{s_k} \det(x^{p-k}A - \lambda I_n)\right)_{|x=0}.$$

Theorem The matrix A is k–irreducible, if and only if the polynomials $\theta_j(\lambda)$, $(j = 1, \ldots, k)$, do not vanish identically in λ.

- Introduced by Hilali and Wazner (1987) as a generalization of Moser Reduction.
- Useful for computing the integer slopes of the Newton polygon and the $\rho-$invariants of Gérard-Levelt.
- First algorithm by HW 1987. Implemented in Maple by Bar-Pfl 1996.
- Barkatou (1997): useful for computing rational solutions of systems with coefficients in $\mathbb{C}(x)$.
- New algorithm by Barkatou-Pflügel 2007: the computation of a super-irreducible form can be reduced to the computation of several Moser-irreducible systems of smaller size.

References

See the abstract of this tutorial in the Proceedings of ISSAC'10.