Symbolic Methods for Solving Systems of Linear Ordinary Differential Equations (I)

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Tutorial at ISSAC'10, Munich, Germany, 25 July 2010

Introduction

Introduction

$\lozenge$ **Main objective:** introduce direct methods for studying matrix linear ordinary differential equations in the complex domain with rational function coefficients.

- Direct methods $\equiv$ methods that do not require prior reduction to a scalar equation.

$\lozenge$ **Main topics:**

- Local Problems: Classification of Singularities, Computing Formal Invariants, Computing Formal Solutions.
- Global Problems: Finding Closed Form Solutions (Polynomial, Rational, Exponential Solutions . . .), Factorization.

$\lozenge$ **General strategy:** Develop and use appropriate tools of local analysis to compute efficiently local data (around singularities).

- Global problems are solved by piecing together the local information around the different singularities.

◊ **Main Focus:** Formal aspects of Local Analysis and Related Computer Algebra Algorithms.

◊ **Not Discussed**: Analytic aspects: Asymptotic and Summability Theory.

▷ This tutorial is divided into three parts:

- Part 1: Basic Tools for Local Analysis - Systems of First Kind. (45 minutes)
- Part 2: Systems of Second Kind - Fundamental Algorithms. (45 minutes)
- Part 3: Applications to Solving Systems with Rational Function Coefficients. (40 minutes)

Part 1: Basic Tools of Local Analysis- Systems of First Kind

Outline

- Equivalent Matrix Linear Differential Equation
- Correspondance Matrix/Scalar Equations
- Classification of Singularities
- Systems of First Kind
- Formal Solutions

Preliminaries

Matrix Linear Differential Equations

We consider a system of first order linear differential equations of the form

$$[A] \qquad\qquad Y' = AY,$$

where Y is column-vector of length n,

A is an $n \times n$ matrix with entries in K,

K is a differential field of characteristic zero with constant field $\mathcal{C} \supset \mathbb{Q}$.

In this talk

$$K = \mathbb{C}((x)) = \mathbb{C}[[x]][x^{-1}], \quad \text{or} \quad K = \mathbb{C}(x)$$

with $' = \frac{d}{dx}$ the standard derivation.

Equivalent Systems

Consider a system $[A] \quad Y' = AY,\ A \in M_n(K)$.

Gauge transformation: $Y = TZ$, $T \in GL(n, K)$, leads to

$$[B] \quad Z' = BZ,$$

$$B = T[A] := T^{-1}AT - T^{-1}T'.$$

Systems $[A]$ and $[B]$ are called equivalent (over K).

If $T \in GL(n, L)$ for some differential field extension L of K then $[A]$ and $[B]$ are called equivalent over L.

Conrrespondance Systems/Equations

Scalar $\longrightarrow$ Matrix

Consider a scalar linear differential equation:

$$D(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0, \quad a_i \in K$$

Let

$$Y = (y, y', \ldots, y^{(n-1)})^T$$

Then

$$Y' = CY$$

where

$$C = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{pmatrix}$$

Notation: $C = companion(a_i)_{0 \leq i \leq n-1}$

Matrix $\longrightarrow$ Scalar

Thm: (Cyclic Vector Lemma) Assume $\exists a \in K, a' \neq 0$. Then every system $Y' = AY$ is equivalent to a scalar equation $D(y) = 0$.

In other words, given a system $Y' = AY$ over K, one can always construct a gauge transformation $T \in GL(n, K)$ such that

$$T[A] := T^{-1}AT - T^{-1}T' = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}$$

• Many proofs : Loewy 1918, Cope 1936, Deligne 1970, Ramis 1978, Katz 1989, Barkatou 1993, Churchill-Kovacic 2002 . . .

Cyclic Vectors

Consider a system $[A]$ $\partial Y = AY$ over a differential field (K, ∂).

Let $\Lambda = (\lambda_1, \cdots, \lambda_n) \in K^n$.

Put

$$y = \Lambda Y = \lambda_1 y_1 + \cdots + \lambda_n y_n$$

Computing successively $\partial y, \cdots, \partial^n y$ and using the equation $\partial Y = AY$ we obtain

$$\partial^i y = \Lambda_i Y \qquad \text{for} \quad i = 0, \cdots, n \tag{1}$$

where the sequence of row vectors $\{\Lambda_i\}$ is defined inductively as:

$$\Lambda_0 = \Lambda, \qquad \Lambda_i = \partial\Lambda_{i-1} + \Lambda_{i-1}A \quad \text{for} \quad i = 1, \cdots, n.$$

Let

$$P = \begin{pmatrix} \Lambda_0 \\ \Lambda_1 \\ \vdots \\ \Lambda_{n-1} \end{pmatrix}, \quad Z = \begin{pmatrix} y \\ \partial y \\ \vdots \\ \partial^{n-1} y \end{pmatrix}, \quad B = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_n \end{pmatrix}.$$

Then (1) can be written as

$$Z = PY \qquad \text{and} \qquad \partial Z = BY \tag{2}$$

Note that $B = \partial(P) + PA$.

Def: The vector Λ is said to be a cyclic vector for system $[A]$ if the matrix P is nonsingular (i.e. $\det P \neq 0$).

If Λ is a cyclic vector then equation (2) can be written

$$Y = P^{-1}Z \qquad \text{and} \qquad \partial Z = CZ$$

$C := BP^{-1} = \partial(P)P^{-1} + PAP^{-1}$ is a companion matrix
$C = companion(a_i)_{0 \leq i \leq n-1}$.

- Hence it follows that the system $\partial Y = AY$ is equivalent to the scalar differential equation :

$$D_\Lambda(y) = \partial^n y + a_{n-1}\partial^{n-1}y + \cdots + a_1\partial y + a_0 y = 0$$

- Note that this equation is by no means uniquely determined by the system $\partial Y = AY$. It depends on the choice of the cyclic vector Λ.
- It is always possible (Cope, Ramis) to choose a cyclic vector Λ whose components are polynomials in x of degree $\leq n - 1$.

Example

Let

$$A := \begin{bmatrix} -x + 2x^{-1} & x^3 + x^2 & 4x^{-1} \\ x^{-3} + 2x^{-2} & 1 - x & 4x^{-1} \\ x & 3x^2 & 2x^{-1} + x^2 \end{bmatrix}$$

Take $\Lambda = [1, 0, 0]$. Then

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{x^2-2}{x} & x^3 + x^2 & 4x^{-1} \\ \frac{2x^2+2+x^4+2x^3+x}{x^2} & -x\left(-6x - 16 + 2x^3 + x^2\right) & 4\,\frac{3-x^2+x^4+2x^3}{x^2} \end{bmatrix}$$

$\det P = -12x + 12x^3 + 4x^5 + 12x^4 - 52 \neq 0$. Hence $\Lambda = [1, 0, 0]$ is a cyclic vector for $[A]$.

Example

Compute $T := P^{-1}$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{-2x^4-2x^3+7x^2-4+2x^5+x+x^6}{x^3(3x^4+3x^3-3x-13+x^5)} & \frac{3-x^2+x^4+2x^3}{x^2(3x^4+3x^3-3x-13+x^5)} & -\frac{1}{x(3x^4+3x^3-3x-13+x^5)} \\ 1/4\,\frac{-14x^3-21x^2+x^5-2x^4+9x+30}{3x^4+3x^3-3x-13+x^5} & 1/4\,\frac{x\left(-6x-16+2x^3+x^2\right)}{3x^4+3x^3-3x-13+x^5} & 1/4\,\frac{x^2(x+1)}{3x^4+3x^3-3x-13+x^5} \end{bmatrix}$$

Compute

$$C := T[A] = T^{-1}AT - T^{-1}T' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_0 & c_1 & c_2 \end{bmatrix}$$

$$c_0 = \frac{-156+269x^4-146x^5-215x^6+38x^9+15x^{10}+16x^8+4x^{11}-109x^7-243x+432x^3}{x^3\left(3x^4+3x^3-3x-13+x^5\right)}$$

$$c_1 = \frac{130-156x^4-24x^5+67x^6+4x^9+2x^{10}+10x^8+42x^7+65x-311x^3-124x^2}{x^2\left(3x^4+3x^3-3x-13+x^5\right)}$$

$$c_2 = \frac{-52+24x^4+6x^5-2x^6+x^8+23x^2+x^7-28x+14x^3}{x\left(3x^4+3x^3-3x-13+x^5\right)}$$

Comments on use of cyclic vectors

♢ **Interest**: Many algorithmic problems are easily solvable for scalar equations.

♢ **Drawbacks**:

- For systems with "large" dimension n (in practice $n \geq 10$), the construction of an equivalent scalar equation may take a "long time".
- The scalar equation (when it can be computed) has often "too complicated" coefficients compared with the entries of the input system (even for small dimensions) and in consequence solving this equation can be costly.

♢ Direct methods are to be preferred.

Solutions- Singularities

Solutions

Consider a system $[A]: \quad Y' = AY, \quad A \in M_n(K)$. We are interested in solutions of $[A]$.

• We should describe the class of functions in which the solutions are to be found:

• Solution over K: a vector $Y \in K^n$ such that $Y' = AY$.

• The set $\mathcal{S}_K = \{Y \in K^n \mid Y' = AY\}$ is a vector-space of dim $\leq n$ over the field of constants of K.

• In general, $\dim \mathcal{S}_K < n$. However, There always exists a differential field extension $K \subset L$ such that over L the solution space has dimension n.

• Fundamental solution matrix of $[A]$: an n by n invertible matrix W (with entries in some extension L of K) satisfying $W' = AW$.

Singularities

Consider a linear differential equation in the complex plane $\mathbb{C}$ with analytic (e.g. meromorphic or rational function) coefficients:

$$[A]: \quad \frac{dY}{dx} = A(x)Y.$$

Def: $x_0 \in \mathbb{C}$ is an ordinary point if all the entries of $A(x)$ are holomorphic in some nbhd of x_0, otherwise x_0 is a singular point.

- x_0 is an ordinary point $\Rightarrow$ there exists a fund soln matrix W whose entries are holomorphic in in some nbhd of x_0.

Classification of Singularities

♢ Suppose that $A(x)$ is holomorphic in a punctured nbhd of x_0, $\Omega = \{x \in \mathbb{C} \,|0 < |x - x_0| < \rho\}$, with at most a pole at the point x_0.

♢ Since Ω is not simply connected, the solutions of Eqn $[A]$ need not be single-valued, but we have the following result (cf. Wasow):

Every fund soln matrix W of $[A]$ has the form:

$$W(x) = \Phi(x)(x - x_0)^{\Lambda}$$

where $\Phi(x)$ is holomorphic on Ω, and Λ is a constant matrix.

Def: The point x_0 is called a regular singular point for $[A]$ if $\Phi(x)$ has at most a pole at the point x_0, otherwise x_0 is called an irregular singular point.

Examples

$y' = \frac{1}{3x} y$ fund soln: $x^{\frac{1}{3}} \Rightarrow x = 0$ is a regular singular point

$y' = \frac{2}{x^3} y$ fund soln: $e^{-\frac{1}{x^2}} \Rightarrow x = 0$ is an irregular singular point.

$Y' = \frac{\Lambda}{x} Y$, $\Lambda \in M_n(\mathbb{C})$, fund soln matrix: $W = x^{\Lambda}$ so $x = 0$ is a regular singular point

$Y' = \frac{\Lambda}{x^2} Y$, $\Lambda \in M_n(\mathbb{C})$, fund soln matrix: $W = \exp\left(\frac{-\Lambda}{x}\right)$

$$x = 0 \text{ is a regular singular point } \iff \Lambda \text{ is a nilpotent matrix.}$$

Classification of the point at ∞

The change of variable $x \mapsto \frac{1}{x}$ permits to classify the point $x = \infty$ as an ordinary, regular singular or irregular singular point for $[A]$:

Let $z = \frac{1}{x}$ $\frac{dY}{dx} = A(x)Y \Rightarrow \frac{dY}{dz} = -\frac{1}{z^2}A(\frac{1}{z})Y$

Type of $x = \infty$ for $\frac{dY}{dx} = A(x)Y$

$$= \text{Type of } z = 0 \text{ for } \frac{dY}{dz} = -\frac{1}{z^2}A(\frac{1}{z})Y$$

Examples

$\frac{dy}{dx} = \frac{1}{3x}y$, $z = \frac{1}{x} \Rightarrow \frac{dy}{dz} = -\frac{1}{3z}y$

$z = 0$ is a regular singular point $\Rightarrow x = \infty$ is a regular singular point

$y' = \frac{2}{x^3}y$ $t = \frac{1}{x} \Rightarrow \frac{dy}{dz} = -2zy$

$z = 0$ is an ordinary point $\Rightarrow x = \infty$ is an ordinary point.

Remarks

• The nature of a singular point x_0 is based upon the knowledge of fundamental solution matrix and hence is not immediately checkable for a given system.

• In the scalar case, the nature of a singular point x_0 can be read off from the leading terms of the coefficients of the equation (Fuchs' Criterion).

• In the matrix case, there is no analogue of the Fuchs' Criterion.

• **Problem:** Give an algorithm to decide for any system whether it has regular singularity.

- First step: Give a characterization of regular singularities which is not based on a prior knowledge of solutions.

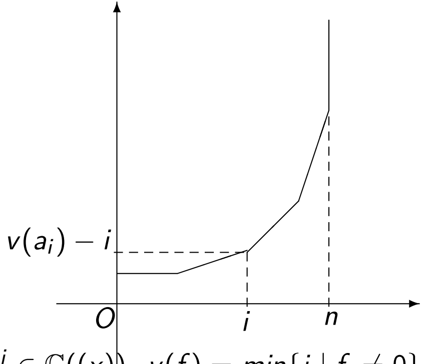
From now we will assume that $x_0 = 0$ (unless otherwise specified).

Newton Polygon of a Scalar Equation

Let

$$D = \sum_{i=0}^{n} a_i \left(\frac{d}{dx}\right)^i, \quad a_i \in \mathbb{C}[[x]].$$

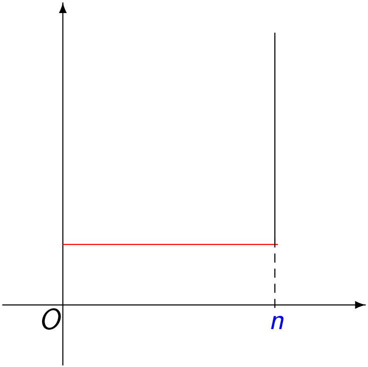
Def: $N(D) :=$ Newton polygon of D = Convex hull with nonnegative slopes of the points $(i, v(a_i) - i)$



$0 \neq f = \sum_j f_j x^j \in \mathbb{C}((x)),\ v(f) = min\{j \mid f_j \neq 0\},\ v(0) = +\infty.$

Fuchs Criterion for Scalar Equations

The point $x = 0$ is regular singular $\iff$ $v(a_i) - i \geq v(a_n) - n$ for all i (Fuchs criterion).



Matrix Case: Another Classification of Singularities

A Classification easier to check

Consider a linear differential system with a singularity at $x = 0$:

$$[A]: \quad Y' = A(x)Y,$$

$$A(x) = \frac{1}{x^{p+1}} \sum_{i=0}^{\infty} A_i x^i, \quad A_i \in M_n(\mathbb{C}), \;\; A_0 \neq 0, \; p \in \mathbb{N}.$$

The integer $p(A) := p$ is called the Poincaré rank of the system $[A]$.

Def:

- If $p(A) = 0$, the point $x_0 = 0$ is called a singularity of first kind for system $[A]$.
- If $p(A) > 0$, the point $x_0 = 0$ is called a singularity of second kind for system $[A]$.

Comparison of the two classifications

◊ The two classifications are not directly comparable. However we have the following (see next section):

- $x = 0$ is a singularity of the first kind $\Rightarrow x = 0$ is a regular singularity.
- The converse is false, in general: a singularity of second kind ($p > 0$) may be a regular singularity.
- System $[A]$ has a regular singularity at $x = 0$ iff $\exists T \in GL(n, K)$ s.t. $T[A]$ has a singularity of first kind at $x = 0$.

◊ **Problem:** Give an algorithm to decide whether a system of second kind has a regular singularity. (Part 2 of this Tutorial)

Systems of First Kind

Systems of First Kind: a Special Case

$$[A]: \; Y' = A(x)Y, \quad A(x) = \frac{A_0}{x} + \sum_{i=1}^{\infty} A_i x^i, \; A_i \in M_n(\mathbb{C}), \; A_0 \neq 0.$$

Thm 1 If the eigenvalues of A_0 do not differ by nonzero integers, then there exists $T \in GL(n, \mathbb{C}[[x]])$ with $T(0) = I_n$ such that $T[A] := T^{-1}AT - T^{-1}T' = \frac{A_0}{x}$

Proof: Look for $T = \sum_{i=0}^{\infty} T_i x^i$ satisfying: $xT' = xAT - TA_0$.

Inserting the series of A and T in the above equation yields:

$$A_0 T_0 - T_0 A_0 = 0, \quad (A_0 - jI_n)T_j - T_j A_0 = -\sum_{i=0}^{j-1} A_{j-i} T_i, \; j \geq 1. \qquad (3)$$

By choosing $T_0 = I_n$, the T_j's are determined recursively by (3) which has a unique solution since $A_0 - jI_n$ and A_0 have no common eigenvalues for $j \geq 1$ (see next slide).

Sylvester Equation

◊ Uniqueness of solution of (3) follows from the following well-known Linear Algebra result :

- Let M and N be two square matrices of order m and n with entries in a field $\mathcal{C}$ and having no common eigenvalues (in the algebraic closure of $\mathcal{C}$). Then for every matrix $L \in Mat_{m,n}(\mathcal{C})$ there exists a unique matrix $X \in Mat_{m,n}(\mathcal{C})$ such that $MX - XN = L$.

◊ The matrix X can be determined by solving a sparse system of mn linear equations in mn unknowns (More efficient algorithms exist).

Systems of First Kind: The General Case

$$[A]:\ Y' = A(x)Y, \quad A(x) = \frac{A_0}{x} + \sum_{i=1}^{\infty} A_i x^i,\ \ A_i \in M_n(\mathbb{C}),\ \ A_0 \neq 0.$$

Thm 2 There exists $T \in GL(n, \mathbb{C}[x])$ with $\det T(x) \neq 0$ for $x \neq 0$ such that $B = T[A] = x^{-1}(B_0 + xB_1 + \dots)$ where the eigenvalues of B_0 do not differ by nonzero integers.

Thm 3 Any system of first kind has a formal fundamental solution matrix of the form

$$\Phi(x)x^{\Lambda}$$

where Λ is a constant matrix and $\Phi \in GL(n, \mathbb{C}((x)))$. Moreover, the formal series $\Phi(x)$ converges whenever the series for $A(x)$ does.

- Many Proofs: Sauvage (1886), ...

Proof of Thm 2.

• Arrange eigenvalues of A_0 in disjoint sets so that the elements in each set differ only by integers.

• Let $\mu_1, \ldots, \mu_s$ the elements of such a set:

$$\Re\mu_1 > \Re\mu_2 > \ldots > \Re\mu_s, \;\; \mu_i - \mu_{i+1} = \ell_i \in \mathbb{N}^*, \;\; i = 1, \ldots, s-1.$$

• Let $\mu_{s+1}, \ldots, \mu_r$ denote the other eigenvalues of A_0.

• For $i = 1, \ldots, r$, denote by m_i the multiplicity of μ_i.

• By applying a constant gauge transformation we can assume that:

$$A_0 = \begin{pmatrix} A_0^{11} & 0 \\ 0 & A_0^{22} \end{pmatrix},$$

where A_0^{11} is an m_1 by m_1 matrix having one single eigenvalue μ_1 :

$$A_0^{11} = \mu_1 I_{m_1} + N_1$$

N_1 being nilpotent matrix.

• Apply the gauge transformation $U = diag(xI_{m_1}, I_{n-m_1})$ yields the new system:

$$Z' = x^{-1}B(x)Z, \quad B(x) = xU^{-1}A(x)U - xU^{-1}U'$$

with the leading matrix:

$$B(0) = \left(A_0 + xU^{-1}A_1U - xU^{-1}U'\right)_{|x=0}.$$

• Let A_1 be partitioned as A_0 :

$$A_1 = \begin{pmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{pmatrix}, \quad A_1^{11} \in \mathbb{C}^{m_1 \times m_1}$$

Then

$$B(0) = \begin{pmatrix} A_0^{11} - I_{m_1} & A_1^{12} \\ 0 & A_0^{22} \end{pmatrix}.$$

Hence the eigenvalues of $B(0)$ are: $\mu_1 - 1, \mu_2, \ldots, \mu_s, \ldots, \mu_r$, each with the same initial multiplicity m_i.

• By repeating this process ℓ_1 times, the eigenvalues become:

$$\mu_1 - \ell_1 = \mu_2, \mu_2, \ldots, \mu_s, \ldots, \mu_r.$$

• Thus, after $\ell_1 + \ldots + \ell_{s-1}$ steps, the eigenvalues $\mu_1, \ldots, \mu_s$ are reduced to one single eigenvalue μ_s of multiplicity $m_1 + \ldots + m_s$.

• By applying the same process to the other groups of eigenvalues, one obtains a matrix B_0 whose eigenvalues do not differ by nonzero integers.

• The matrix T in Theorem 2 is obtained as a product of invertible constant matrices or matrices of type U.
Hence T is a polynomial matrix with $\det T(x) = cx^\nu$ for some $c \in \mathbb{C}$ and $\nu \in \mathbb{N}^*$.

Example

$$A(x) = \begin{pmatrix} 2x^{-1} - 2 & -2 + x^{-1} & 0 \\ 2 & 2x^{-1} + 2 & x \\ 3 & 3 & x^{-1} \end{pmatrix} \quad A_0 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Put $U = diag(x, x, 1)$ and let $B := xU[A]$

$$B(x) = xU^{-1}AU - xU^{-1}U' = \begin{pmatrix} -2x + 1 & -2x + 1 & 0 \\ 2x & 1 + 2x & x \\ 3x^2 & 3x^2 & 1 \end{pmatrix}.$$

$$B_0 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has 1 as eigenvalue of multiplicity 3.

Formal Solutions of a Singular Differential System

Formal Solutions

A differential equation

$$Y' = AY, \quad A \in M_n(\mathbb{C}[[x]][x^{-1}])$$

has a formal fundamental solution matrix of the form

$$\Phi(x^{1/s})x^{\Lambda} \exp\left(Q(x^{-1/s})\right)$$

$s \in \mathbb{N}^*$, $\Phi \in GL(n, \mathbb{C}[[x^{1/s}]])$,
$Q(x^{-1/s}) = \mathrm{diag}\left(q_1(x^{-1/s}), \ldots, q_n(x^{-1/s})\right)$

the q_i's are polynomials in $x^{-1/s}$
Λ is a constant matrix commuting with Q.

Exponential Part

- Q is invariant under gauge transformations $T \in GL(n, \overline{\mathbb{C}((x))})$.
- $x = 0$ is regular singular $\iff Q(x^{-1/s}) \equiv 0$. In this case $s = 1$ and the formal series $\Phi(x)$ converges whenever the series for $A(x)$ does.
- When $Q(x^{-1/s}) \not\equiv 0$, the origin is an irregular singular point of the system. In this case $\Phi(x)$ need not be convergent even if the series for $A(x)$ does.
- The elements of $Q(x^{-1/s})$ determine the main asymptotic behavior of actual solutions as $x \to 0$ in sectors of sufficiently small angular opening (Asymptotic Existence Theorem (cf. Wasow)).

- Existence :
  - Matrix Case: Hukuhara (1930's), Turritin (1950's) , Wasow (1960's), Levelt , Jurkat, Lutz, Balser (1970's) , Babbit& Varadajan (1980's), ...
  - Scalar Case: Fabry (1885), Poincaré (1886), Malgrange (1978), ...
- Algorithms (for related problems):
  - Matrix Case: Moser (1960's), Dietrich (1970's), Levelt, Wagenfuhrer, Hilali&Wazner (1980's), Chen , Barkatou, (1990's), Pflügel (2000), Corel (2003), Barkatou (2004), Barkatou-Pflügel (2007)
  - Scalar Case: Frobenius method- Newton Algorithm : Della-Dora et al. (1986), Dietrich (1986), Barkatou (1988), ...

Implementation

- Scalar Equations:
  - DESIR: in REDUCE by J. Della Dora, E. Tournier, C. Dicrescenzo (1986)
  - ELISE: in Maple by V. Dietrich (1980's)
  - DESIR 2: in Maple by Barkatou, E. Pflügel (1996)
  - DEtools Package of Maple: M. v. Hoeij (1996)
- Matrix Equations:
  - ISOLDE: in Maple by Barkatou, E. Pflügel (1996). Available at: http://sourceforge.net/projects/isolde
  - LFS Package of Maple : Abramov et al.

Examples

- Euler Equation: $x^2y' + y = x$

It has a formal power series solution $\hat{f} = \sum_{n=0}^{\infty}(-1)^n n! x^{n+1}$

Homogeneous Equation: $x^3y'' + (x^2 + x)y' - y = 0$

$$\Rightarrow \frac{dY}{dx} = \begin{pmatrix} 0 & 1 \\ \frac{1}{x^3} & -(\frac{1}{x} + \frac{1}{x^2}) \end{pmatrix} Y, \quad Y = \begin{pmatrix} y \\ y' \end{pmatrix}$$

Formal Solution Matrix: $Y = \Phi(x)e^Q$

$$Q = \begin{pmatrix} \frac{1}{x} & 0 \\ 0 & 0 \end{pmatrix} \quad \Phi(x) = \begin{pmatrix} 1 & \hat{f} \\ -\frac{1}{x^2} & \hat{f}' \end{pmatrix}$$

- Airy Equation: $y'' = xy$

$x = \infty$ is a singular point at ∞

$z = \frac{1}{x} \Rightarrow z^5 y'' + 2z^4 y' - y = 0$

$$\Rightarrow Y' = \begin{pmatrix} 0 & 1 \\ \frac{1}{z^5} & -\frac{2}{z} \end{pmatrix} Y$$

Formal Solution Matrix: $Y = \Phi(z) U z^J U^{-1} e^{Q(t)}$

$$\Phi(z) = \dots, \quad U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{3}{4} \end{pmatrix}$$

$$t = z^{\frac{1}{2}}, \quad Q = \begin{pmatrix} -\frac{2}{3t^3} & 0 \\ 0 & \frac{2}{3t^3} \end{pmatrix}$$

Some Textbooks

W. Balser,
Formal power series and linear systems of meromorphic ordinary differential equations
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References

See the abstract of this tutorial in the Proceedings of ISSAC'10.